國立宜蘭大學 101 學年度微積分競試試題解答

1.Which statement is false?

(A) $e=\lim_{n\to\infty}\left[1+\frac{1}{n}\right]^n$ (B) $e=\lim_{t\to 0}[1+t]^{1/t}$ (C) $\lim_{n\to\infty}\left[1+\frac{x}{n}\right]^n=e^x$ (D) $\lim_{t\to 0}[1+xt]^{1/xt}=e^x$ (E) $\lim_{t\to 0}[1+xt]^{1/t}=e^x$

2. Evaluate $\lim_{x\to 0}\frac{|x|}{x}=$

(A)-1 (B)0 (C)1 (D)2 (E)limit doesn't exist Ans:(E)

$$\lim_{x\to 0}\frac{|x|}{x}\Rightarrow\begin{cases}\lim_{x\to 0^+}\frac{|x|}{x}=\lim_{x\to 0^+}\frac{x}{x}=1\\ \lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=-1\end{cases}\Rightarrow \text{limit doesn't exist}$$

3. Evaluate $\lim_{x\to 0}\frac{\sin 7x}{\sin 5x}=$

(A) $\frac{7}{5}$ (B) $\frac{5}{7}$ (C) 0 (D) $\frac{5}{3}$ (E) $\frac{3}{5}$ Ans:(A)

$$\lim_{x\to 0}\frac{\sin 7x}{\sin 5x}=\lim_{x\to 0}\frac{7\frac{\sin 7x}{7x}}{5\frac{\sin 5x}{5x}}=\frac{\lim_{x\to 0}7\frac{\sin 7x}{7x}}{\lim_{x\to 0}5\frac{\sin 5x}{5x}}=\frac{7}{5}$$

4. Let $f(u,v)=u^v$, where $u=\tan x$ and $v=\sin x$, please find $\frac{df}{dx}$ (Express the final answer in x only.).

(A) $\sin x\cdot\tan x$ (B) $\cos x\cdot\sec^2 x$ (C) $\sin x\cdot(\tan x)^{\sin x-1}\cdot\sec^2 x$ (D) $(\tan x)^{\sin x}\left[\frac{1}{\cos x}+\cos x\cdot\ln\tan x\right]$

(E) $(\tan x)^{\sin x}\cdot\cos x[1+\ln\tan x]$

$$\frac{df}{dx}=\frac{\partial f}{\partial u}\cdot\frac{du}{dx}+\frac{\partial f}{\partial v}\cdot\frac{dv}{dx}=vu^{v-1}\cdot\frac{1}{\cos^2 x}+u^v\ln u\cdot\cos x=u^v\left[\frac{v}{u}\cdot\frac{1}{\cos^2 x}+\ln u\cdot\cos x\right]=(\tan x)^{\sin x}\left[\frac{1}{\cos x}+\cos x\cdot\ln\tan x\right]$$

5. Determine whether Rolle's Theorem can be applied to $f(x)=\frac{x^2-14}{x}$ on the closed interval $[-14,14]$. If Rolle's Theorem can be applied, find all values of c in the open interval $(-14,14)$ such that $f'(c)=0$

(A)c=1, c=10 (B)c=10 (C)c=4,c=1 (D)c=4 (E) Rolle's Theorem does not apply Ans:(E)

$f(a)\neq f(b)$, Rolle's Theorem does not apply.

6. Assume $y=f(x)$ is a continuous function, and satisfies $y=x^{x+y}$,please find $y'(1)$

(A)-2(B) -1(C) 0(D) 1(E)2 Ans:(E)

Note that if $x=1$, then $y=1$ since $y=1^{1+y}=1$. Taking derivative after taking ln on both side yields

$\ln y = (x+y)\ln x$

$\frac{y'}{y} = (1+y)\ln x + (x+y)\frac{1}{x}$

Hence $y' = \dfrac{\frac{xy+y^2}{x}\, y\ln x}{1-y\ln x}$ and $y'(1) = 2$

7. Evaluate $\int_0^1 x^5 e^{x^3} dx$

(A) -1 (B) $-\frac{1}{3}$ (C) 0 (D) $\frac{1}{3}$ (E) 1 Ans:(D)

Let: $u = x^3$

$$\int_0^1 x^5 e^{x^3} dx = \int_0^1 u \cdot \frac{1}{3} \cdot e^u du = \frac{1}{3}\left[ue^u\Big|_0^1 - \int_0^1 e^u du\right] = \frac{1}{3}$$

8. If $f(x)$ is a continuous function defined in R, and $\int_0^{x^2} f(t)dt = x\sin\pi x$, then $f(1) = ?$

(A) 1 (B) $\frac{-\pi}{2}$ (C) 0 (D) $\frac{\pi}{2}$ (E) -2 Ans:(B)

$$\int_0^{x^2} f(t)dt = x\sin\pi x$$

$$\Rightarrow \frac{d}{dx}\int_0^{x^2} f(t)dt = \frac{d}{dx}(x\sin\pi x)$$

$$\Rightarrow f(x^2)\cdot 2x = \sin(\pi x) + \pi x\cos\pi x$$

$$\Rightarrow 2f(1) = -\pi$$

$$\Rightarrow f(1) = \frac{-\pi}{2}$$

9.Assume $F(x) = \int_0^{x^2} \sin\theta^2 d\theta$, please find the $F'(x)$.

(A) $2x\sin x^4$ (B) $2x\sin x^2$ (C) $2x\cos x^4$ (D) $2x\cos x^2$ (E) $\sin x^4 \cdot \cos x^4$ Ans:(A)

$$\frac{d}{dx}\int_{p(x)}^{q(x)} f(t)dt = f(q(x))\cdot q'(x) - f(p(x))\cdot p'(x)$$

$$\Rightarrow \frac{d}{dx}\int_0^{x^2} \sin\theta^2 d\theta = \sin\left[(x^2)^2\right]\cdot 2x - 0 = 2x\sin x^4$$

10. Evaluate $\int_0^1 \frac{x^3-1}{\ln x} dx = ?$

(A) ln 3 (B) 2 ln 3 (C)2ln 2 (D) 3ln 2 (E) 3ln 3 Ans:(C)

$$\int_0^1 \frac{x^3-1}{\ln x}dx = \int_0^1\int_0^3 x^y dydx = \int_0^3\int_0^1 x^y dxdy = \int_0^3\left[\frac{1}{y+1}x^{y+1}\right]_0^1 dy = \int_0^3 \frac{1}{y+1}dy = [\ln y+1]_0^3 = \ln 4 - \ln 0 = \ln 2^2 = 2\ln 2$$

11. Evaluate $\int \csc x dx = ? + C$ (C is constant)

(A) $\ln\left|\frac{\sin x}{1+\cos x}\right|$ (B) $\ln|\csc x+\cot x|$ (C) $\sec x$ (D) $-\sec x$ (E) no solution Ans:(A)

$$\int \csc x dx = \int \csc x \cdot \frac{(\csc x+\cot x)}{(\csc x+\cot x)}dx = -\int \frac{-\csc^2 x-\csc x\cot x}{(\csc x+\cot x)}dx = -\ln|\csc x+\cot x|+C = \ln\left|\frac{1}{\csc x+\cot x}\right|+C$$

$$= \ln\left|\frac{1}{\frac{1}{\sin x}+\frac{\cos x}{\sin x}}\right|+C = \ln\left|\frac{\sin x}{1+\cos x}\right|+C$$

12.試問，此題中從哪個步驟起開始計算錯誤 which of the following step is misevaluated

$$\int_{-\infty}^{\infty} x^3 dx \overset{(A)}{=} \lim_{t\to\infty}\int_{-t}^{t} x^3 dx \overset{(B)}{=} \lim_{t\to\infty}\frac{1}{4}x^4\Big|_{-t}^{t} \overset{(C)}{=} 0$$

(D)答案為 0 無誤 (E)無從判定 Ans:(A)

$$\int_{-\infty}^{\infty} x^3 dx = \int_{-\infty}^{0} x^3 dx + \int_{0}^{\infty} x^3 dx = \lim_{s\to-\infty}\int_{s}^{0} x^3 dx + \lim_{t\to\infty}\int_{0}^{t} x^3 dx$$

Consider $\lim_{t\to\infty}\int_0^t x^3 dx = \lim_{t\to\infty}\frac{1}{4}x^4\Big|_0^t = \lim_{t\to\infty}\frac{1}{4}t^4 = \infty \Rightarrow$ diverge

$\therefore \int_{-\infty}^{\infty} x^3 dx$ diverge

13. Evaluate $\int \frac{1}{a^2+x^2}dx = ?+C$ (C is constant)

(A) $\frac{1}{a}\cdot\tan\frac{x}{a}$ (B) $sec^{-1}x$ (C) $\frac{1}{a}\cdot\sin^{-1}x$ (D) $\frac{1}{a}\cdot\cos^{-1}x$ (E) $\frac{1}{a}\cdot\tan^{-1}\frac{x}{a}$ Ans:(E)

$Let: x = a\tan\theta \quad dx = a\sec^2\theta d\theta$

$$\therefore \int \frac{a\sec^2\theta}{a^2+a^2\tan^2\theta}d\theta = \int \frac{a\sec^2\theta}{a^2(1+\tan^2\theta)}d\theta = \frac{\theta}{a}+C = \frac{1}{a}\cdot\tan^{-1}\frac{x}{a}+C$$ (C is constant)

14. Evaluate $\int \frac{1}{a^4+x^4}dx = ?+C$ (C is constant)

(A) $\frac{1}{2a^3}\left[\ln|a+x|+\ln|a-x|+2\tan^{-1}\frac{x}{a}\right]$ (B) $\frac{1}{4a^3}\left[\ln|a+x|-\ln|a-x|+2\tan^{-1}\frac{x}{a}\right]$

(C) $\frac{3}{4a^3}\left[\ln|a+x|-\ln|a-x|+\frac{1}{a}\tan^{-1}\frac{x}{a}\right]$ (D) $\frac{x}{4a^3}\left[\ln|a+x|-\ln|a-x|+\frac{x}{a}\tan^{-1}\frac{x}{a}\right]$

(E) $\frac{1}{4a^3}\left[\ln|2a+x|+\ln|a-x|+\tan^{-1}\frac{x}{a}\right]$ Ans:(B)

$$\int \frac{1}{a^4+x^4}dx = \int\left[\frac{1}{4a^3}\left(\frac{1}{a-x}+\frac{1}{a+x}\right)+\frac{1}{2a^2}\cdot\frac{1}{a^2+x^2}\right]dx$$

$$= \frac{1}{4a^3}\int\left[\frac{1}{a-x}+\frac{1}{a+x}+\frac{2}{a^2+x^2}\right]dx = \frac{1}{4a^3}\left[\ln|a+x|-\ln|a-x|+2\tan^{-1}\frac{x}{z}\right]+C$$ (C is constant)

15. If $f(x)=\sqrt{x^3+x+6}$, find $(f^{-1})'(4)$

(A) $\frac{3}{7}$ (B) $\frac{4}{9}$ (C) $\frac{5}{11}$ (D) $\frac{8}{13}$ (E) $\frac{6}{15}$ Ans:(D)

$$4=\sqrt{x^3+x+6}\Rightarrow x=2$$

$$\therefore (f^{-1})'(4)=\frac{1}{f'(2)}=\frac{1}{\frac{13}{8}}=\frac{8}{13}$$

16. Evaluate $\int_1^5 \frac{\ln x}{x^3}dx=?$

(A) $\frac{1872}{625}$ (B) $\frac{-1}{3}\left[\frac{2\ln 7+1}{50}-\frac{1}{2}\right]$ (C) $\frac{24-2\ln 5}{100}$ (D) $\frac{-1}{2}\left[\frac{2\ln 5+1}{50}-\frac{1}{4}\right]$ (E) $\frac{1}{2}\left[\frac{2\ln 5+1}{50}-\frac{1}{2}\right]$

$$Let: u=\ln x \Leftrightarrow x=e^u$$

$$du=\frac{1}{x}dx$$

$$\int_1^5 \frac{\ln x}{x^3}dx=\int_1^5 \frac{1}{x}\cdot\frac{\ln x}{x^2}dx=\int_0^{\ln 5}\frac{u}{e^{2u}}du=\int_0^{\ln 5}u\cdot e^{-2u}du=\frac{-1}{2}\int_0^{\ln 5}u\cdot de^{-2u}=\frac{-1}{2}\left[u\cdot e^{-2u}-\int e^{-2u}du\right]_0^{\ln 5}$$

$$=\frac{-1}{2}\left[u\cdot e^{-2u}+\frac{e^{-2u}}{2}\right]_0^{\ln 5}=\frac{-1}{2}\left[\frac{2u+1}{2e^{2u}}\right]_0^{\ln 5}=\frac{-1}{2}\left[\frac{2\ln x+1}{2x^2}\right]_1^5=\frac{-1}{2}\left[\frac{2\ln 5+1}{50}-\frac{1}{2}\right]=\frac{24-2\ln 5}{100}$$

17. Considering a surface: $f(x,y)=x^2+3xy+y^2+2$, please find the equation of tangent plane at point $(1,2,1)$.

(A) $7x+5y-3z=18$ (B) $8x+7y-z=21$ (C) $x+2y+z=15$ (D) $5x+5y+5z=25$ (E) $x+2y-3z=4$

Ans:(B)

$$z=x^2+3xy+y^2+2$$

$$F(x,y,z)=x^2+3xy+y^2-z+2=0$$

$$\nabla f(x,y,z)=<2x+3y,3x+2y,-1>$$

$$\nabla f(1,2,1)=<8,7,-1>$$

$$\text{tangent plane}:<x-1,y-2,z-1>\cdot<8,7,-1>=0$$

$$\Rightarrow \underline{8x+7y-z=21}$$

18. If we know $\sinh x=\frac{e^x-e^{-x}}{2}$, please evaluate $\int_0^{\ln 2}\tanh x dx=?$

(A) $\ln\left(\frac{5}{2}\right)$ (B) $\ln\left(\frac{5}{3}\right)$ (C) $\ln\left(\frac{5}{4}\right)$ (D) $\ln\left(\frac{5}{6}\right)$ (E) $\ln\left(\frac{5}{7}\right)$

$$\int_0^{\ln 2}\tanh x dx=\int_0^{\ln 2}\frac{\sinh x}{\cosh x}dx=\left[\ln(\cosh x)\right]_0^{\ln 2}=\ln(\cosh(\ln 2))-\ln(\cosh(0))=\ln\left(\frac{5}{4}\right)-0=\ln\left(\frac{5}{4}\right)$$

19.Evaluate $\int \frac{\cosh x}{\sqrt{9-\sinh^2 x}}dx=?$

(A) $\arcsin\left(\frac{e^x-e^{-x}}{2}\right)+C$ (B) $\arcsin\left(\frac{e^x-e^{-x}}{6}\right)+C$ (C) $\arccos\left(\frac{e^x-e^{-x}}{6}\right)+C$ (D) $\arcsin\left(\frac{e^x+e^{-x}}{6}\right)+C$

(E) $\arcsin\left(\dfrac{e^x+e^{-x}}{2}\right)+C$

$Let: u=\sinh x \quad du=\cosh x dx$

$$\int \frac{\cosh x}{\sqrt{9-\sinh^2 x}}dx=\arcsin\left(\frac{\sinh x}{3}\right)+C=\arcsin\left(\frac{e^x-e^{-x}}{6}\right)+C$$

20. Find the curvature κ of the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{1}=1$ at x=0.

(A)0 (B) 2 (C)$\dfrac{1}{2}$ (D) 4 (E)$\dfrac{1}{4}$ Ans:(C)

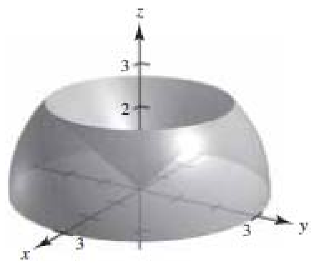
$$x^2+4y^2=4$$
$$\Rightarrow y=\frac{1}{2}\left(\sqrt{2^2-x^2}\right)$$
$$\Rightarrow y'=\frac{1}{2}\times\frac{-x}{\sqrt{2^2-x^2}}$$
$$\Rightarrow y''=\frac{1}{2}\times\frac{2^2}{\left(2^2-x^2\right)^{3/2}}$$
$$x=0\Rightarrow\begin{cases}y'=0\\ y''=\dfrac{1}{4}\end{cases}$$
$$\kappa=\frac{|y''|}{\left[1+(y')^2\right]^{3/2}}=\frac{1}{4}$$

21. Use the definition of Taylor series to find the Taylor series for function $f(x)=\ln(x^2+1)$, centered at $c=0$

(A)$\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{n+2}}{2n+1}$ (B)$\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+2}}{n+1}$ (C)$\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+3}}{2n+1}$ (D)$\sum_{n=0}^{\infty}\dfrac{(-1)^{n+1} x^{2n+2}}{3n+1}$ (E)$\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+3}}{n+2}$ Ans:(B)

$$\ln(x^2+1)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!}=0+0x+\frac{2x^2}{2!}+\frac{0x^3}{3!}-\frac{12x^4}{4!}+\frac{0x^5}{5!}+\frac{240x^6}{6!}+\ldots=x^2-\frac{x^4}{2}+\frac{x^6}{3}-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}$$

22. Use spherical coordinates to find the volume of the solid. Solid inside $x^2+y^2+z^2=9$, outside $z=\sqrt{x^2+y^2}$, and above the xy-plane.



(A)$9\pi\sqrt{2}$ (B)$9\pi\sqrt{5}$ (C)$7\pi\sqrt{3}$ (D)$7\pi\sqrt{5}$ (E)$7\pi\sqrt{11}$ Ans:(A)

$$V=\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^3\rho^2\sin\phi\cdot d\rho\cdot d\phi\cdot d\theta=\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}9\sin\phi\cdot d\phi\cdot d\theta$$
$$=\int_0^{2\pi}\left[-9\cos\phi\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta=\int_0^{2\pi}9\left(\frac{\sqrt{2}}{2}\right)d\theta=18\pi\left(\frac{\sqrt{2}}{2}\right)=9\pi\sqrt{2}$$

23. Assume that Plane A is parallel to Plane B and they are flying at the same height. If Plane B, 200 kilometers away from Plane A in the southwest, is flying towards the south at 40 kilometers per minutes and Plane B is

flying towards the east at 20 kilometers per minutes, when will it occur that the distance between Plane A and Plane B is the shortest?

(A) $3\sqrt{3}$ (B) $2\sqrt{3}$ (C) $2\sqrt{5}$ (D) $3\sqrt{5}$ (E) $3\sqrt{2}$ Ans:(E)

24. According to question 23, what is the minimum in kilometers?

(A) $17\sqrt{2}$ (B) $15\sqrt{3}$ (C) $20\sqrt{10}$ (D) $25\sqrt{2}$ (E) $18\sqrt{10}$ Ans:(C)

$Let: f(t)=\left(\frac{200}{\sqrt{2}}-40t\right)^2+\left(\frac{200}{\sqrt{2}}-20t\right)^2$ =兩機在 t 分鐘後距離平方

$f'(t)=0 \Rightarrow t=3\sqrt{2}\min$ ，且 $f'(t)$ 在 $t=3\sqrt{2}$ 處變號

$\therefore$ 在 $t=3\sqrt{2}$ 時兩機距離最小，其距離= $f(3\sqrt{2})=20\sqrt{10}km$

25.Evaluate $\int\frac{-1}{4x-x^2}dx=?$

(A) $\frac{1}{2}\ln\left|\frac{x-4}{x}\right|+C$ (B) $\frac{1}{3}\ln\left|\frac{x-4}{x}\right|+C$ (C) $\frac{1}{5}\ln\left|\frac{x-16}{x}\right|+C$ (D) $\frac{1}{4}\ln\left|\frac{x-4}{x}\right|+C$ (E) $\frac{1}{3}\ln\left|\frac{x-8}{2x}\right|+C$

$\int\frac{-1}{4x-x^2}dx=\int\frac{1}{(x-2)^2-4}dx=\frac{1}{4}\ln\left|\frac{(x-2)-2}{(x-2)+2}\right|+C=\frac{1}{4}\ln\left|\frac{x-4}{x}\right|+C$