國立宜蘭大學 102 學年度第一學期微積分競試試題題解

第一部分:單選題 15 題，每題答對得 4 分，答錯或選項多於一個者倒扣一分，未作答不予計分。

1. Evaluate $\lim_{x\to\infty}\dfrac{x-\cos x}{x^3\tan^2\left(\dfrac{2}{x}\right)}=?$

(A) -1 (B) 0 (C) $\frac{1}{2}$ (D) $\frac{1}{4}$ (E) 1

Ans:

$$\lim_{x\to\infty}\frac{x-\cos x}{x^3\tan^2\left(\frac{2}{x}\right)}=\lim_{x\to\infty}\left(\frac{\frac{2}{x}}{\tan\left(\frac{2}{x}\right)}\right)^2\times\frac{1}{4}\times\frac{x-\cos x}{x}=1\times\frac{1}{4}\times 1=\frac{1}{4}$$

By squeeze theorem:

$$\lim_{x\to\infty}\frac{x-\cos x}{x}\Rightarrow\frac{1+x}{x}\ge\frac{x-\cos x}{x}\ge\frac{x-1}{x}$$

$$\lim_{x\to\infty}\frac{1+x}{x}=1=\lim_{x\to\infty}\frac{x-1}{x}\qquad\therefore\lim_{x\to\infty}\frac{x-\cos}{x}=1$$

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2. Let $f(x)=\dfrac{e^{9x^2+100}\sqrt{3x^2-1}}{x^3+5x-3}$, find $f'(x)$=?

(A) $e^{9x^2+100}\cdot\left(\dfrac{6x}{3x^2-1}-\dfrac{3x^2+5}{x^3+5x-3}\right)$

(B) $e^{9x^2+100}\cdot\left(4x+\dfrac{4x}{3x^2-1}+\dfrac{x^2+4}{3x^3+5x-3}\right)$

(C) $\dfrac{e^{9x^2+100}\sqrt{3x^2-1}}{x^3+5x-3}\left(18x+\dfrac{3x}{3x^2-1}-\dfrac{3x^2+5}{x^3+5x-3}\right)$

(D) $\dfrac{e^{9x^2+100}\cdot(3x^2-1)^{\frac{3}{2}}}{(x^3+5x-3)^2}\left(\dfrac{5x}{3x^2-1}+\dfrac{8x}{5x^2+4}\right)$

(E) $\dfrac{e^{9x^2+100}\cdot(3x^2-1)^2}{2(2x^3+5x-3)}\left(6x+\dfrac{3x^2-7}{4x^3+5x-1}\right)$

Ans:

Let $y=\dfrac{e^{9x^2+100}\sqrt{3x^2-1}}{x^3+5x-3}$

$$y=\frac{e^{9x^2+100}\sqrt{3x^2-1}}{x^3+5x-3}\Rightarrow \ln y=\left(9x^2+100\right)+\frac{1}{2}\ln\left(3x^2-1\right)-\ln\left(x^3+5x-3\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=18x+\frac{1}{2}\cdot\frac{6x}{3x^2-1}-\frac{3x^2+5}{x^3+5x-3}$$

$$\Rightarrow \frac{dy}{dx}=\frac{e^{9x^2+100}\sqrt{3x^2-1}}{x^3+5x-3}\left(18x+\frac{3x}{3x^2-1}-\frac{3x^2+5}{x^3+5x-3}\right)$$

3. Suppose that $f(0)=-3$ and $f'(x)\le 5$ for all values of x. How large can $f(2)$ possibly be ?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Ans:

By mean value theorem :

$$f'(x)=\frac{f(2)-f(0)}{2-0}=\frac{f(2)+3}{2}\le 5\Rightarrow f(2)\le 7$$

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4. Evaluate $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}(\tan x)^{\cos x}=?$

(A) 0 (B) 1 (C) $\frac{\sqrt{2}}{2}$ (D) $\frac{1}{2}$ (E) not exist

Ans:

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-}(\tan x)^{\cos x}=\lim_{x\to\left(\frac{\pi}{2}\right)^-}e^{\cos x\ln(\tan x)}=e^{\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\ln(\tan x)}{\sec x}}$$

$$=e^{\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\frac{1}{\tan x}\cdot\sec^2 x}{\sec x\cdot\tan x}}=e^{\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\cos x}{\sin^2 x}}=e^0=1$$

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5. Evaluate $\lim\limits_{x\to 3}\frac{x}{x-3}\int_3^x\frac{\sin t}{t}dt=?$

(A) $\sin 3$ (B) $\sin 9$ (C) $\cos 3$ (D) $\cos 9$ (E) $\tan 9$

Ans:

$$\lim_{x\to 3}\frac{x\int_3^x\frac{\sin t}{t}dt}{x-3}=\lim_{x\to 3}\frac{1\cdot\int_3^x\frac{\sin t}{t}dt+x\cdot\frac{\sin x}{x}}{1}=0+\sin 3=\sin 3$$

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6. Evaluate $\int x\sin^{-1}x dx=?$ (答案選項應為 A，但因設計錯誤，故此題送分。)

(A) $\frac{1}{2}\left(x^2\cdot\sin^{-1}x+\frac{1}{2}x\sqrt{1-x^2}\right)+c$ (B) $\frac{1}{4}\left(\sin^{-1}x-2x\sin^{-1}x+\sqrt{1-x^2}\right)+c$

(C) $\frac{1}{8}\left(x^2\sin^{-1}x-\frac{1}{2}\sin^{-1}x\cdot 3x+x^2\sqrt{1-x^2}\right)+c$ (D) $\frac{3}{4}\left(x\sin^{-1}x-2x\sin^{-1}x+x\sqrt{1-x^2}\right)+c$

(E) $\frac{1}{2}x^2\sin^{-1}x-\frac{1}{4}x\cdot\sin^{-1}x+2x\sqrt{1-x^2}+c$

Ans:

$$\int x\sin^{-1}x\,dx=\int\sin^{-1}x\,d\left(\frac{1}{2}x^2\right)=\sin^{-1}x\cdot\frac{1}{2}x^2-\int\frac{1}{2}x^2\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$$=\frac{1}{2}\sin^{-1}x\cdot x^2-\frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}dx=\frac{1}{2}\left(\sin^{-1}x\cdot x^2-\frac{1}{2}\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}\right)+c$$

Let $x=\sin\theta$ $\quad dx=\cos\theta d\theta$

$$\int\frac{x^2}{\sqrt{1-x^2}}dx=\int\frac{\sin^2\theta}{\cos\theta}\cos\theta d\theta=\int\sin^2\theta d\theta=\int\frac{1-\cos 2\theta}{2}d\theta$$

$$=\frac{1}{2}\theta-\frac{1}{2}\cdot\frac{1}{2}\sin 2\theta+c=\frac{1}{2}\sin^{-1}x-\frac{1}{4}x\sqrt{1+x^2}+c$$

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7. Evaluate $\int_2^\infty\frac{2x^3+2x^2+x-1}{x^2(x-1)(x^2+1)}dx=?$

(A) $\frac{\pi}{4}+\ln 5-\tan^{-1}4$ (B) $\frac{\pi}{2}+4-\ln 2+\sin^{-1}3$ (C) $\frac{\sqrt{2}}{2}+\frac{1}{2}+\ln 4-\sin^{-1}2$

(D) $\frac{\sqrt{2}}{2}+1+\ln 5-\sin^{-1}4$ (E) $\frac{\pi}{2}+\frac{1}{2}+\ln 5-\tan^{-1}2$

Ans:

$$\int_2^\infty\frac{2x^3+2x^2+x-1}{x^2(x-1)(x^2+1)}dx=\int_2^\infty\left(\frac{1}{x^2}+\frac{2}{x-1}-\frac{2x+1}{x^2+1}\right)dx$$

$$=\int_2^\infty\frac{1}{x^2}dx+\int_2^\infty\frac{2}{x-1}dx-\int_2^\infty\frac{2x}{x^2+1}dx+\int_2^\infty\frac{1}{x^2+1}dx$$

$$=\lim_{t\to\infty}\left[\frac{-1}{x}+2\ln(x-1)-\ln(x^2+1)+\tan^{-1}x\right]\Big|_2^t$$

$$=\lim_{t\to\infty}\left[\frac{1}{x}+\ln\left(\frac{(x-1)^2}{x^2+1}\right)+\tan^{-1}x\right]\Big|_2^t=\frac{\pi}{2}+\frac{1}{2}+\ln 5-\tan^{-1}2$$

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8. Which of the following series is convergent ?

(A) $\sum_{n=1}^{\infty} n\ln\left(1+\frac{1}{n}\right)$ (B) $\sum_{n=1}^{\infty}\frac{n!}{n3^n}$ (C) $\sum_{n=1}^{\infty}\frac{1}{1+\ln n}$

(D) $\sum_{n=1}^{\infty}\sin\frac{1}{n}$ (E) $\sum_{n=1}^{\infty}2^n\sin\left(\frac{\pi}{3^n}\right)$

Ans:

(A) $\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n=\ln e=1\neq 0$ $\therefore$divergent.

(B) $\lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)\cdot 3^{n+1}}}{\frac{n!}{n\cdot 3^n}}=\lim_{n\to\infty}\frac{(n+1)\cdot n}{(n+1)\cdot 3}=\frac{n}{3}>1$ $\therefore$divergent.

(C) $1+2\ln n<1+2n\le n+2n=3n$, $\therefore\frac{1}{1+2\ln n}>\frac{1}{3n}$

$\because\sum_{n=1}^{\infty}\frac{1}{3n}=\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n}$ divergent $\therefore\sum_{n=1}^{\infty}\frac{1}{1+2\ln n}$ $\therefore$divergent

(D) $\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}=1$ $\because\sum_{n=1}^{\infty}\frac{1}{n}$ divergent $\therefore\sum_{n=1}^{\infty}\sin\frac{1}{n}$ divergent

(E) $\lim_{n\to\infty}\sqrt[n]{\frac{\sin\left(\frac{\pi}{3^n}\right)}{\frac{\pi}{3^n}}\cdot 2^n\cdot\left(\frac{\pi}{3^n}\right)}=\frac{2}{3}<1$ $\therefore$convergent

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9. Evaluate $\lim_{(x,y)\to(0,0)}\frac{xy^2}{x^4+y^2}=?$

(A) -1 (B) 0 (C) 1 (D) $\frac{1}{2}$ (E) Does not exist

Ans:

Let $x^2=r\cos\theta, y=r\sin\theta$

$$\lim_{(x,y)\to(0,0)}\frac{xy^2}{x^4+y^2}=\lim_{r\to 0}\frac{\sqrt{r\cos\theta}\cdot r^2\sin^2\theta}{r^2}=\lim_{r\to 0}\sqrt{r\cos\theta}\cdot\sin^2\theta=0$$

10. Let $f(x,y,z)=x^{y^z}$, find $\frac{\partial f}{\partial z}$ =?

(A) $x^{y^z} \cdot y \ln x \cdot y^z \cdot \ln y$ (B) $x^{y^z} \cdot \ln x \cdot z \cdot y^{z-1}$

(C) $x^{y^z} \cdot \ln x \cdot y^z \cdot \ln y$ (D) $x^{y^z} \cdot \ln x \cdot z \cdot y^{z-1}$ (E) $x^{y^z} \cdot y \ln x \cdot z \cdot y^{z-1}$

Ans:

Let $g = x^{y^z}$

$$\Rightarrow \ln g = y^z \cdot \ln x \Rightarrow \frac{1}{g}\frac{\partial g}{\partial z} = y^z \cdot \ln y \cdot \ln x \Rightarrow \frac{\partial g}{\partial z} = x^{y^z} \cdot y^z \cdot \ln y \cdot \ln x$$

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11. Find the value of $\frac{\partial x}{\partial z}$ at point (1,-1,3) if the equation $xz + y\ln x - x^2 + 4 = 0$.

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{1}{6}$ (D) $\frac{1}{8}$ (E) 1

Ans:

Let $F(x, y, z) = xz + y\ln x - x^2 + 4$

$$\left.\frac{\partial x}{\partial z}\right|_{(1,-1,-3)} = -\left.\frac{Fy}{Fz}\right|_{(1,-1,-3)} = -\left.\frac{x}{z + \frac{y}{x} - 2x}\right|_{(1,-1,-3)} = \frac{1}{6}$$ (答案選項應為C，但因題目設計錯誤，故此題送分。)

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12. Consider $f(x, y, z) = x^2 y + y^3 z + xz^3$ at the point (2,1,-1) .Find the maximum rate of change of $f(x, y, z)$ =?

(A) $\sqrt{31}$ (B) $\sqrt{37}$ (C) $\sqrt{47}$ (D) $\sqrt{59}$ (E) $\sqrt{61}$

Ans:

$$\nabla f = \left(2xy + z^3, x^2 + 3y^2 z, y^3 + 3xz^2\right)$$
$$\nabla f = (2,1,-1) = (3,1,7) \Rightarrow |\nabla f| = \sqrt{3^2 + 1^2 + 7^2} = \sqrt{59}$$

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13. If $x^2 + y^2 + z^2 = 48$, find the maximum of xyz =?

(A) 64 (B) 88 (C) 100 (D) 120 (E) 168

Ans:

$$L(x,y,z)=xyz+\lambda\left(x^2+y^2+z^2-48\right)$$

$$\begin{cases}\dfrac{\partial L}{\partial x}=yz+2x\lambda=0\\ \dfrac{\partial L}{\partial y}=xz+2y\lambda=0\\ \dfrac{\partial L}{\partial z}=xy+2z\lambda=0\\ \dfrac{\partial L}{\partial \lambda}=x^2+y^2+z^2-48=0\end{cases} \Rightarrow (x,y,z)=(4,4,4) \therefore xyz=64$$

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14. Evaluate the integral $\int_0^2\int_0^{4-x^2}\dfrac{xe^{2y}}{4-y}dydx=?$

(A) e^2-1 (B) $\dfrac{3}{4}\left(e^2-2\right)$ (C) e^6-4 (D) $\dfrac{1}{6}\left(e^4-2\right)$ (E) $\dfrac{1}{4}\left(e^8-1\right)$

Ans:

$$\int_0^2\int_0^{4-x^2}\frac{xe^{2y}}{4-y}dydx=\int_0^2\int_0^{\sqrt{4-y}}\frac{xe^{2y}}{4-y}dxdy=\int_0^4\frac{1}{2}(4-y)\frac{e^{2y}}{4-y}dy$$

$$=\frac{1}{2}\int_0^4 e^{2y}dy=\frac{1}{4}e^{2y}\Big|_0^4=\frac{1}{4}\left(e^8-1\right)$$

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15. Evaluate the integral $\iint_R e^{\frac{x+y}{x-y}}dxdy$, where R is the trapezoidal region with vertices (1,0) , (2,0) , (0,-2) and (0,-1) .

(A) $\dfrac{1}{2}(e-1)$ (B) $\dfrac{3}{4}\left(e+e^{-1}\right)$ (C) $\dfrac{3}{2}\left(e^2-e\right)$ (D) $\dfrac{2}{3}\left(e^2-e^{-1}\right)$ (E) $\dfrac{1}{5}\left(e^2-e-1\right)$

Ans:

Let $\begin{cases}u=x+y\\ v=x-y\end{cases}\Rightarrow\begin{cases}x=\dfrac{u+v}{2}\\ y=\dfrac{u-v}{2}\end{cases}\Rightarrow |J|=\dfrac{1}{2}$

$$\iint_R e^{\frac{x+y}{x+y}}dxdy=\int_1^2\int_{-v}^{v}e^{\frac{u}{v}}\cdot\frac{1}{2}dudv=\frac{1}{2}\int_1^2\left(v\cdot e^{\frac{u}{v}}\right)\Bigg|_{-v}^{v}dv$$

$$=\frac{1}{2}\left(e-e^{-1}\right)\int_1^2 vdv=\frac{1}{2}\left(e-e^{-1}\right)\left(\frac{1}{2}v^2\right)\Bigg|_1^2=\frac{3}{4}\left(e-e^{-1}\right)$$

第二部分:非選題兩題，每題各兩小題，共計40分，請將答案用藍黑色原子筆寫在答案紙上。

1. Given a function f which satisfies the differential equation

$$xf''(x)+3x\left[f'(x)\right]^2=1-e^{-x}$$

for all real x .(Do not attempt to solve this differential equation).

(a) If f has an extremum at a point $c\neq 0$, show that this extremum is a minimum.

(b) If f has an extremum at 0, is it a maximum or a minimum? Justify your conclusion.

Ans:

$(a) cf''(c)+3c\left[f'(c)\right]^2=1-e^{-c}$ $\because f'(c)=0, \therefore f''(c)=\dfrac{1-e^{-c}}{c}, c\neq 0$

if $c>0,\ f''(c)>0$

if $c<0,\ f''(c)>0$ $\therefore$ relative minimun

(b) $f''(x)+3\left[f'(x)\right]^2=\dfrac{1-e^{-x}}{x}$

$\lim\limits_{x\to 0} f''(x)=\lim\limits_{x\to 0}\dfrac{1-e^{-x}}{x}=\lim\limits_{x\to 0}\dfrac{e^{-x}}{1}=1>0$ $\therefore$ relative minimun

2. (a) Let u be a nonzero solution of the second-order equation

$$y''+P(x)y'+Q(x)y=0$$

Show that the substitution $y=uv$ converts the equation

$$y''+P(x)y'+Q(x)y=R(x)$$

into a first-order linear equation for v'.

(b) Obtain a nonzero solution of the equation $y''-4y'+x^2(y'-4y)=0$ by inspection and use the method of part (a) to find a solution of

$$y''-4y'+x^2(y'-4y)=2xe^{-x^3/3}$$

such that $y=0$ and $y'=0$ when $x=0$.

(a)

$$y'' + P(x)y' + Q(x)y = 0$$
$$y = uv$$
$$y' = u'v + uv'$$
$$y'' = u''v + u'v' + u'v' + uv'' = u''v + 2u'v' + uv''$$
$$\Rightarrow (u''v + 2u'v' + uv'') + P(x)(u'v + uv') + Q(x)(uv) = 0$$
$$\Rightarrow v(u'' + P(x)u' + Q(x)u) + uv'' + (Pu + 2u')v' = R(x)$$
$$v' = \phi, \phi' = v'', Pu + 2u' = \delta$$
$$\Rightarrow u\phi' + \delta\phi = R(x)$$

(b)

$$y'' - 4y' + x^2(y' - 4y) = 0, \because y' - 4y = 0 \therefore u = e^{4x}$$
$$\Rightarrow y'' - 4y' + x^2(y' - 4y) = 2xe^{-x^3/3}$$
$$y = ve^{4x}$$
$$y' = v'e^{4x} + 4ve^{4x}$$
$$y'' = v''e^{4x} + 4v'e^{4x} + 4v'e^{4x} + 16ve^{4x}$$
$$\Rightarrow (v''e^{4x} + 4v'e^{4x} + 4v'e^{4x} + 16ve^{4x}) - 4(v'e^{4x} + 4ve^{4x}) + x^2(v'e^{4x} + 4ve^{4x} - 4ve^{4x}) = 2xe^{-x^3/3}$$
$$\Rightarrow v''e^{4x} + 4v'e^{4x} + x^2v'e^{4x} = 2xe^{-x^3/3}$$
$$\Rightarrow v'' + 4v' + x^2v' = 2xe^{-x^3/3-4x}$$
$$\phi = v', \phi' = v'', \phi' + (x^2 + 4)\phi = 2xe^{-x^3/3-4x}$$
$$I.F = e^{\int(x^2+4)dx} = e^{1/3x^3+4x}$$
$$\Rightarrow (e^{x^3/3+4x} \cdot \phi)' = 2x$$
$$e^{x^3/3+4x} \cdot \phi = x^2 + c_1$$
$$\Rightarrow \phi = v' = (x^2 + c_1)e^{-x^3/3-4x}$$
$$v = \int^x (x^2 + c_1)e^{-x^3/3-4x}dx + c_2$$
$$y = ve^{4x} = e^{4x}\int^x (x^2 + c_1)e^{-x^3/3-4x}dx + c_2e^{4x}$$

Ans: $y(0) = 0 = c_2$

$$y'(0) = 4$$
$$y' = v'e^{4x} + 4ve^{4x} = (x^2 + c_1)e^{-x^3/3} + 4e^{4x}\int^x (x^2 + c_1)e^{-x^3/3-4x}dx$$
$$y'(0) = 4 = c_1$$
$$y = e^{4x}\int^x (x^2 + 4)e^{-x^3/3-4x}dx = -(e^{-x^3/3} - e^{4x})$$