國立宜蘭大學 103 學年度第一學期微積分競試試題題解

1.Assume $f(x,y)=\ln\sqrt[4]{x^2+y^2}$，please find $f_{xx}+f_{yy}=?$

(A)0　(B)$(x^2+y^2)^{-2}$　(C)$2(x^2+y^2)^{-2}$　(D)$\frac{1}{2}(x^2+y^2)^{-2}$

$$f(x,y)=\ln\sqrt[4]{x^2+y^2}=\frac{1}{4}\ln x^2+y^2$$
$$f_x(x,y)=\frac{1}{4}\times\frac{2x}{x^2+y^2}=\frac{x}{2(x^2+y^2)}$$
$$f_{xx}(x,y)=\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}=\frac{-x^2+y^2}{2(x^2+y^2)}$$
$$f_y(x,y)=\frac{1}{4}\times\frac{2y}{x^2+y^2}=\frac{y}{2(x^2+y^2)}$$
$$f_{yy}(x,y)=\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}=\frac{x^2-y^2}{2(x^2+y^2)}$$
$$f_{xx}+f_{yy}=0$$

2.Assume $f(x)=\frac{x(x-1)(x-2)(x-3)....(x-n)}{(x+1)(x+2)(x+3)....(x+n)}$，please find $f'(0)=?$

(A)0　(B)1　(C)-1　(D)$(-1)^n$

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}$$
$$=\lim_{x\to 0}\frac{f(x)}{x}$$
$$=\lim_{x\to 0}\frac{(x-1)(x-2)(x-3)....(x-n)}{(x+1)(x+2)(x+3)....(x+n)}=(-1)^n$$

3.For what interval of **x** does the series $\sum_{k=1}^{\infty}\frac{7^k}{k!}x^k$ converge?

(A)(-1,1)　(B)(-7,7)　(C)$(-\infty,\infty)$　(D) not exist

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$$

$$\lim_{k\to\infty}\frac{\frac{7^{n+1}}{(n+1)!}x^{n+1}}{\frac{7^n}{n!}x^n}=\lim_{n\to\infty}\frac{7}{n+1}x=0<1 \quad \text{恆成立}$$

$(-\infty,\infty)$

4. $\int_0^{\sqrt{3}}\frac{1}{(1+x)\sqrt{x}}dx=?$

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{2}$ (D) π

$$\int_0^{\sqrt{3}}\frac{1}{(1+x)\sqrt{x}}dx \quad \left(\text{令}\sqrt{x}=y,\ x=y^2\ dx=2ydy\right)$$

$$=\int_0^{\sqrt{3}}\frac{2y}{(1+y^2)y}dy$$

$$=\int_0^{\sqrt{3}}\frac{2}{(1+y^2)}dy$$

$$=2\arctan y\Big|_0^{\sqrt{3}}=\frac{2}{3}\pi$$

5. $\int_0^{\frac{\pi}{2}}\frac{1}{\sin x+\cos x}dx=?$

(A) $\sqrt{2}\ln\left(\sqrt{2}+1\right)$ (B) $\sqrt{2}\ln\left(\sqrt{2}-1\right)$ (C) $\sqrt{2}\ln\left(\sqrt{2}+1\right)^2$ (D) $\frac{1}{\sqrt{2}}\ln\left(\sqrt{2}+1\right)$

$$\int_0^{\frac{\pi}{2}}\frac{1}{\sin\ x+\cos\ x}$$

$$=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{1}{\sin\ x\cos\ \frac{\pi}{4}+\cos\ x\sin\ \frac{\pi}{4}}dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{1}{\sin(\ x+\frac{\pi}{4})}dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\csc(\ x+\frac{\pi}{4})dx$$

$$=\frac{1}{\sqrt{2}}\ln\left|\csc(\ x+\frac{\pi}{4})-\cot(\ x+\frac{\pi}{4})\right|\Bigg|_0^{\frac{\pi}{2}}$$

$$=\frac{1}{\sqrt{2}}\left[\ln\left(\sqrt{2}+1\right)-\ln\left(\sqrt{2}-1\right)\right]=\sqrt{2}\ln\left(\sqrt{2}+1\right)$$

6.Let $f:R\to R$ be a twice continuously differentiable function, $f(0)=1, f(1)=2\ and\ f'(1)=0$ .Compute $\int_0^1 xf''(x)dx$.

(A)-1 (B)0 (C)1 (D)2

$$\int_0^1 xf''(x)dx$$
$$(let: u=x\ \ du=dx\ \ dv=f''(x)dx\ \ v=f'(x)$$
$$=\left[xf'(x)-\int f'(x)dx\right]_0^1$$
$$=\left[xf'(x)-f(x)\right]_0^1=2-2+1=1$$

7.Evaluate $\int_0^1\int_{\sqrt{x}}^1 \sin y^3 dydx$.

(A) $\frac{1}{3}\cos 1-\frac{1}{3}$ (B) $\frac{1}{3}\sin 1-\frac{1}{3}$ (C) $-\frac{1}{3}\cos 1+\frac{1}{3}$ (D) $-\frac{1}{3}\sin 1+\frac{1}{3}$

$$\int_0^1\int_{\sqrt{x}}^1 \sin y^3 dydx=\int_0^1\int_0^{y^2} \sin y^3 dxdy$$
$$=\int_0^1 y^2\sin y^3 dy=-\frac{1}{3}\cos y^3\Big|_0^1=-\frac{1}{3}\cos 1+\frac{1}{3}$$

8.Assume $f(x,y)=\int_x^{y^2} e^{2t}dt+\int_{x^3}^{y}\cos t^2 dx=0$, please find $\frac{dy}{dx}$.

(A) $\frac{e^{2x}+3x^2\cos x^6}{ye^{2y}+\cos y^2}$ (B) $\frac{e^{2x}-3x^2\cos x^6}{2ye^{2y}+\cos y^2}$

(C) $\frac{-e^{2x}-3x^2\cos x^6}{2ye^{2y}+\cos y^2}$ (D) $\frac{e^{2x}+3x^2\cos x^6}{2ye^{2y}+\cos y^2}$

$$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{-e^{2x}-\cos x^6\left(3x^2\right)}{e^{2y^2}\left(2y\right)+\cos y^2}=\frac{e^{2x}+3x^2\cos x^6}{2ye^{2y}+\cos y^2}$$

9. $\int_0^1\int_0^{\sqrt{1-x^2}}\left(x^2+y^2\right)^{-\frac{1}{2}}dxdy=?$

(A) π (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

$$\int_0^1\int_0^{\sqrt{1-X^2}}\left(x^2+y^2\right)^{-\frac{1}{2}}dxdy=\int_0^{\frac{\pi}{2}}\int_0^1\left(r^2\right)^{-\frac{1}{2}}rdrd\theta$$
$$\Rightarrow\int_0^{\frac{\pi}{2}}\int_0^1\left(r^2\right)^{-\frac{1}{2}}rdrd\theta$$
$$=\int_0^{\frac{\pi}{2}}d\theta=\frac{\pi}{2}$$

10. Evaluate $\sum_{n=1}^{\infty}\ln\left[\frac{(n+1)^2}{n(n+2)}\right]$.

(A) Not exist (B)0 (C) $\ln 2$ (D) $\ln 3$

$$\sum_{n=1}^{\infty}\ln\left[\frac{(n+1)^2}{n(n+2)}\right]=\sum_{n=1}^{\infty}\ln\frac{n+1}{n}+\sum_{n=1}^{\infty}\ln\frac{n+1}{n+2}$$

$$\Rightarrow\sum_{n=1}^{\infty}\ln\frac{n+1}{n}+\sum_{n=1}^{\infty}\ln\frac{n+1}{n+2}$$

$$=\left[\ln\frac{2}{1}+\ln\frac{3}{2}+\ln\frac{4}{3}+...\right]+\left[\ln\frac{2}{3}+\ln\frac{3}{4}+\ln\frac{4}{5}+...\right]=\ln 1+\ln 2=\ln 2$$

11. Evaluate $\int \sin^2 x \cot^3 x dx$.

(A) $\ln\cos x-\frac{1}{2}\cos^2 x+C$ (B) $\ln\sin x-\frac{1}{2}\sin^2 x+C$

(C) $\ln\cos x-\frac{1}{2}\sin^2 x+C$ (D) $\ln\sin x-\frac{1}{2}\cos^2 x+C$

$$\int \sin^2 x\cot^3 x dx=\int \sin^2 x\frac{\cos^3 x}{\sin^3 x}dx$$

$$\Rightarrow\int \sin^2 x\frac{\cos^3 x}{\sin^3 x}dx$$

$$=\int\frac{\cos^3 x}{\sin x}dx$$

$$=\int\frac{\cos^2 x}{\sin x}\cos x dx$$

$$=\int\frac{1-\sin^2 x}{\sin x}\cos x dx \qquad (let\ u=\sin x\ \ du=\cos x dx)$$

$$=\int\frac{1-u^2}{u}du=\ln u-\frac{1}{2}u^2+C=\ln\sin x-\frac{1}{2}\sin^2 x+C$$

12. Find $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$.

(A) 1 (B) $e^{\frac{1}{2}}$ (C) $e^{\frac{1}{3}}$ (D) $e^{\frac{1}{4}}$

$$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} \Rightarrow e^{\lim_{x\to 0}\frac{\ln\tan x-\ln x}{x^2}}$$

$$\lim_{x\to 0}\frac{\ln\tan x-\ln x}{x^2}$$

by L'Hopital's Rule

$$=\lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x}-\frac{1}{x}}{2x}$$

$$=\lim_{x\to 0}\frac{\frac{1}{\sin x\cos x}-\frac{1}{x}}{2x}$$

$$=\lim_{x\to 0}\frac{x-\sin x\cos x}{2x^2\sin x\cos x}$$

by L'Hopital's Rule

$$=\lim_{x\to 0}\frac{1-\left(\cos^2 x-\sin^2 x\right)}{4x\sin x\cos x+2x^2\cos^2 x-2x^2\sin x}$$

$$=\lim_{x\to 0}\frac{2\sin^2 x}{4x\sin x\cos x+2x^2\cos^2 x-2x^2\sin x}$$

$$=\lim_{x\to 0}\frac{1}{2\frac{x}{\sin x}\cos x+\frac{x^2}{\sin^2 x}\cos^2 x-\frac{2x^2}{\sin x}}=\frac{1}{2+1-0}=\frac{1}{3}$$

13. $\int_4^5\frac{3x-7}{(x-1)(x-2)(x-3)}dx=?$

(A) $4\ln 3+4\ln 2$ (B) $3\ln 3+4\ln 2$ (C) $4\ln 3-4\ln 2$ (D) $3\ln 3-4\ln 2$

$$\int_4^5\frac{3x-7}{(x-1)(x-2)(x-3)}dx=\int_4^5\frac{-2}{(x-1)}+\frac{1}{x-2}+\frac{1}{x-3}dx$$

$$\Rightarrow\int_4^5\frac{-2}{(x-1)}+\frac{1}{x-2}+\frac{1}{x-3}dx$$

$$=\ln\left|\frac{(x-2)(x-3)}{(x-1)^2}\right|_4^5$$

$$=\ln 3-3\ln 2-\ln 2+2\ln 3=3\ln 3-4\ln 2$$

14. Consider $f(x,y,z)=x^3y+2y^2z^2+x^2z^3$ at the point (2,1,-1) .Find the maximum rate of change of $f(x,y,z)$.

(A) $4\sqrt{17}$ (B) $4\sqrt{21}$ (C) $4\sqrt{37}$ (D) $4\sqrt{59}$

$$\nabla f=\left(3x^2y+2xz^3,x^3+4yz^2,4y^2z+3x^2z^2\right)$$

$$\nabla f(2,1,-1)=(8,12,8)\Rightarrow|\nabla f|=4\sqrt{2^2+3^2+2^2}=4\sqrt{17}$$

15. Assume $f(\frac{1+x}{1-x})=x$, please find $f'(x)=?$

(A) $\frac{1+x}{1-x}$ (B) $\frac{1}{x}$ (C) $\frac{2}{(1+x)^2}$ (D) 1

$$y=\frac{1+x}{1-x}\Rightarrow x=\frac{y-1}{y+1}=1-\frac{2}{y+1}$$

$$f(y)=1-\frac{2}{y+1}$$

$$\Rightarrow f'(y)=\frac{2}{(y+1)^2}$$

16. Which of the following series is divergent?

(A) $\sum_{n=1}^{\infty}\sin\frac{1}{n}$ (B) $\sum_{n=1}^{\infty}\frac{\tan^{-1}n}{n^2+1}$ (C) $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ (D) $\sum_{n=1}^{\infty}\frac{1}{n(\ln n)^s}$ $(s>1)$

$$(A)\because \lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=\lim_{y\to 0^+}\frac{\sin y}{y}=1 \therefore \lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=1$$

$$\because \sum_{n=1}^{\infty}\frac{1}{n} \text{ is divergent } \because \sum_{n=1}^{\infty}\sin\frac{1}{n} \text{ is divergent}$$

$$(B)\because 0<\frac{\tan^{-1}n}{n^2+1}<\frac{\frac{\pi}{2}}{n^2+1}<\frac{2}{n^2} \quad \sum_{n=1}^{\infty}\frac{2}{n^2} \text{ is convergent}$$

$$\because \sum_{n=1}^{\infty}\frac{\tan^{-1}n}{n^2+1} \text{ is convergent}$$

$$(C)\because \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!}{n!}\times\frac{n^n}{n!}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{-1}=\frac{1}{e}<1$$

$$\therefore \sum_{n=1}^{\infty}\frac{n!}{n^n} \text{ is convergent}$$

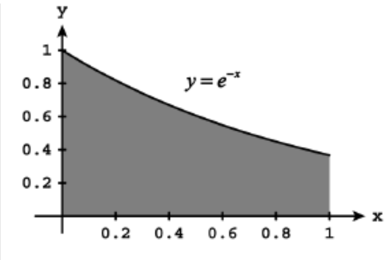
$$(D) \text{let } f(x)=\frac{1}{x(\ln x)^s},\ x\in[2,\infty),\ s>1$$

$$\because \int_2^{\infty}\frac{1}{x(\ln x)^s}dx=\lim_{b\to\infty}\int_2^{b}\frac{1}{x(\ln x)^s}dx=\lim_{b\to\infty}\left[\frac{1}{-s+1}(\ln x)^{-s+1}\right]_2^b=\frac{1}{s-1}(\ln 2)^{1-s}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n(\ln n)^s}\ (s>1) \text{ is convergent}$$

17. The region enclosed by $y=e^{-x}$, $x=0$ and $x=e$ is rotated about the line $y=0$. Find the volume of the resulting solid.

(A) $\dfrac{\pi(e^2-1)}{e^2}$ (B) $\dfrac{2\pi(e^2-1)}{3e^2}$ (C) $\dfrac{\pi(e^2+1)}{e^2}$ (D) $\dfrac{\pi(e^2-1)}{2e^2}$



$R(x)=e^{-x}$

$\Rightarrow V=\int_0^1 \pi[R(x)]^2\,dx$

$=\int_0^1 \pi[e^{-x}]^2\,dx$

$=\pi\int_0^1 e^{-2x}dx=-\dfrac{\pi}{2}e^{-2x}\Big|_0^1=\dfrac{\pi(e^2-1)}{2e^2}$

18. Assume $e^z+x+y+z=0$, please find $\dfrac{\partial^2 z}{\partial x^2}=?$

(A) $-e^z(1+e^z)^{-2}$ (B) $-e^z(1+e^z)^{-3}$ (C) $e^z(1+e^z)^{-2}$ (D) $e^z(1+e^z)^{-3}$

$e^z+x+y+1=0\Rightarrow(1+e^z)\dfrac{\partial z}{\partial x}+1=0$

$\Rightarrow\dfrac{\partial z}{\partial x}=-(1+e^z)^{-1}$

$\Rightarrow\dfrac{\partial^2 z}{\partial x^2}=e^z(1+e^z)^{-2}\dfrac{\partial z}{\partial x}=-e^z(1+e^z)^{-3}$

19. Let $f(x,y)=x^{2^y}$, find $\dfrac{\partial f}{\partial y}$.

(A) $x^{2^y}\cdot 2\ln x\cdot 2^y\cdot\ln 2$ (B) $x^{2^y}\cdot\ln x\cdot y\cdot 2^{y-1}$

(C) $x^{2^y}\cdot 2^y\cdot\ln 2\cdot\ln x$ (D) $x^{2^y}\cdot\ln x\cdot 2^{y-1}$

let $g=x^{2^y}$

$\Rightarrow\ln g=2^y\cdot\ln x\Rightarrow\dfrac{1}{g}\dfrac{\partial g}{\partial y}=2^y\cdot\ln 2\cdot\ln x\Rightarrow\dfrac{\partial g}{\partial y}=x^{2^y}\cdot 2^y\cdot\ln 2\cdot\ln x$

20. Suppose that $f(x)=\begin{cases}3k+\sqrt{x} & 0\le x\le 4\\ 2kx-7x & 4<x\le 9\end{cases}$ is continuous on [0,9]. Then ***k***=?

(A)6 (B)7 (C)8 (D)9

$\lim_{x\to 4^+} f(x)=\lim_{x\to 4^-} f(x)$

$\Rightarrow\begin{cases}\lim_{x\to 4^-} 3k+\sqrt{x}=3k+2\\ \lim_{x\to 4^+} 2kx-7x=8k-28\end{cases}$

$3k+2=8k-28\Rightarrow k=6$

第二部分:非選題，20分，請將答案用藍黑色原子筆寫在答案紙上。

1. Assume human population size (人口數量大小) x at time t can be presented as

$$x = \frac{M}{1+\exp(-\alpha(t-t_0))}$$, where α and the maximal population size M are positive constants and t_0 is the time at which $x = \frac{1}{2}M$. Suppose a census (人口調查) is taken at three equally spaced time t_1, t_2 and t_3, the resulting numbers being x_1, x_2 and x_3. Show that

$$M = x_2 \frac{x_3(x_2-x_1)-x_1(x_3-x_2)}{x_2{}^2 - x_1x_3}.$$

$$\alpha = kM$$

$$x_1 = \frac{M}{1+e^{-\alpha(t_1-t_0)}} \Rightarrow \frac{M}{x_1} = 1+e^{-\alpha(t_1-t_0)} \Rightarrow \frac{M}{x_1} - 1 = e^{-\alpha(t_1-t_0)}$$

$$x_2 = \frac{M}{1+e^{-\alpha(t_2-t_0)}} \Rightarrow \frac{M}{x_2} = 1+e^{-\alpha(t_2-t_0)} \Rightarrow \frac{M}{x_2} - 1 = e^{-\alpha(t_2-t_0)}$$

$$x_3 = \frac{M}{1+e^{-\alpha(t_3-t_0)}} \Rightarrow \frac{M}{x_3} = 1+e^{-\alpha(t_3-t_0)} \Rightarrow \frac{M}{x_3} - 1 = e^{-\alpha(t_3-t_0)}$$

$$t_1 - t_2 = t_2 - t_3$$

$$\frac{\frac{M}{x_1}-1}{\frac{M}{x_2}-1} = \frac{e^{-\alpha(t_1-t_0)}}{e^{-\alpha(t_2-t_0)}} = e^{-\alpha(t_1-t_2)} = \frac{\frac{M}{x_2}-1}{\frac{M}{x_3}-1} = \frac{e^{-\alpha(t_2-t_0)}}{e^{-\alpha(t_3-t_0)}} = e^{-\alpha(t_2-t_3)}$$

$$\left(\frac{M}{x_2}-1\right)^2 = \left(\frac{M}{x_1}-1\right)\left(\frac{M}{x_3}-1\right)$$

$$\Rightarrow \left(\frac{M}{x_2}\right)^2 - 2\frac{M}{x_2} + 1 = \frac{M^2}{x_1x_3} - \left(\frac{M}{x_1}+\frac{M}{x_3}\right) + 1$$

$$\Rightarrow \frac{M}{x_2^2} - \frac{2}{x_2} = \frac{M}{x_1x_3} - \left(\frac{1}{x_1}+\frac{1}{x_3}\right)$$

$$\Rightarrow M\left(\frac{1}{x_2^2} - \frac{1}{x_1x_3}\right) = \frac{2}{x_2} - \frac{1}{x_1} - \frac{1}{x_3}$$

$$\Rightarrow M = \frac{\frac{2x_1x_3 - x_2x_3 - x_1x_2}{x_1x_2x_3}}{\frac{x_1x_3 - x_2}{x_1x_2x_3}} = x_2\frac{x_3(x_1-x_2) - x_1(x_2-x_3)}{x_1x_3 - x_2^2}$$