105學年度第一学期微積分一期末考

1. 求 $\frac{dy}{dx}$

(1) $y = 5x^{e+1} + 2$ $\qquad \frac{dy}{dx} = 5(e+1)x^{e}$

(2) $y = 3e^{x^6}$ $\qquad \frac{dy}{dx} = 18x^5 e^{x^6}$

(3) $y = \ln\sqrt{x+2} + x$ $\qquad \frac{dy}{dx} = \frac{1}{2(x+2)} + 1$

$y = \frac{1}{2}\ln(x+2) + x$

(4) $y = \frac{1}{\ln x}$ $\qquad \frac{dy}{dx} = -(\ln x)^{-2} \cdot \frac{1}{x}$

$= (\ln x)^{-1}$ $\qquad = -\frac{1}{x(\ln x)^2}$

(5) $y = \ln\sqrt{\frac{1+x^2}{1-x^2}}$ $\qquad \frac{dy}{dx} = \frac{1}{2} \cdot \frac{\frac{4x(1-x^2) - 2x^2(-2x)}{(1-x^2)^2}}{\frac{1+x^2}{1-x^2}}$

$= \frac{1}{2}\ln\left(1 + \frac{2x^2}{1-x^2}\right)$

$$= \frac{4x - 4x^3 + 4x^3}{2(1-x^2)(1+x^2)}$$

$$= \frac{2x}{(1-x^2)(1+x^2)}$$

$$= \frac{2x}{1-x^4}$$

(6) $y = x^x$

$\ln y = x \ln x$ $\qquad \frac{dy}{dx} = x^x(1 + \ln x)$

$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$

(7) $y = \dfrac{(x-1)^2\sqrt{2x-3}}{(x+4)^2(2x-1)^4}$

$\ln y = 2\ln(x-1) + \frac{1}{2}\ln(2x-3) - 2\ln(x+4) - 4\ln(2x-1)$

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x-1} + \frac{1}{2x-3} - \frac{2}{x+4} - \frac{8}{2x-1}$

$\frac{dy}{dx} = \left(\frac{2}{x-1} + \frac{1}{2x-3} - \frac{2}{x+4} - \frac{8}{2x-1}\right)\cdot \frac{(x-1)^2\sqrt{2x-3}}{(x+4)^2(2x-1)^4}$

(8) $x = \frac{1}{\sqrt{1+y^2}}$

$x = (1+y^2)^{-\frac{1}{2}}$

$\frac{dx}{dy} = -\frac{1}{2}(1+y^2)^{-\frac{3}{2}}(2y)$

$\therefore \frac{dx}{dy} = -y(1+y^2)^{-\frac{3}{2}}$

$\frac{dy}{dx} = -\frac{(1+y^2)^{\frac{3}{2}}}{y}$

另解： $x^2 = (1+y^2)^{-1}$

$2x = -(1+y^2)^{-2}(2y)\frac{dy}{dx}$

$x = -y(1+y^2)^{-2}\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{x}{y(1+y^2)^{-2}}$

$\frac{dy}{dx} = \frac{-2(1-y^2)}{2xy}$

$\frac{dy}{dx} = -\frac{1+y^2}{xy}$

(9) $x^{\ln x} = x^2y + 2$

$(\ln x)\ln x = \ln(2+x^2y)$

$\frac{2\ln x}{x} = \frac{x^2\frac{dy}{dx} + 2xy}{2+x^2y}$

$2\ln x(2+x^2y) = x^3\cdot\frac{dy}{dx} + 2x^2y$

$\frac{dy}{dx} = \frac{2(2+x^2y)\cdot\ln x - 2x^2y}{x^3}$

$\frac{dy}{dx} = \frac{(\frac{2}{x}\ln x)(x^2y+2) - 2xy}{x^2}$

2

(10) $y = x^{e^x}$

$\ln y = e^x \cdot \ln x$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}e^x + \ln x \cdot e^x$

$\frac{dy}{dx} = (x^{e^x})(e^x)\left(\frac{1+x\ln x}{x}\right)$

2. $y = e \cdot \sqrt{x^3+6}$

$\frac{dy}{dx} = \frac{1}{2}e(x^3+6)^{-\frac{1}{2}}(3x^2)$

$= \frac{3}{2}ex^2(x^3+6)^{-\frac{1}{2}}$

3. $f(x) = \frac{2}{x+1} = 2(x+1)^{-1}$

$f'(x) = -2(x+1)^{-2}$

$f''(x) = 4(x+1)^{-3}$

$\vdots$

$f^n(x) = 2n!(x+1)^{-(n+1)}$

$f'(1) = -2 \cdot 2^{-2} = -\frac{1}{2}$

$f''(1) = \frac{1}{2}$

$f^n(x) = (-1)^n 2n!(x+1)^{-(n+1)}$

4. $xe^y - 10x + 3y = 0$

$e^y + xe^y\frac{dy}{dx} - 10 + 3\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{10-e^y}{3+xe^y}$

5 $f^{(7)}(x) = (x^2-1)^2$

$f^{(8)}(x) = 2(x^2-1)(2x) = 4x(x^2-1) = 4(x^3-x)$

$f^{(9)}(x) = 4(3x^2-1) = 12x^2-4$

$f^{(10)}(x) = 24x$

6. $y = \frac{\ln x}{x} = (x^{-1})\ln x$

$\frac{dy}{dx} = -x^{-2}\cdot\ln x + x^{-2}$

$= x^{-2}(1-\ln x) = \frac{1-\ln x}{x^2}$

(1) ~~臨界點：(1. 0)~~ $\ln x = 1$. $x = e$

(2) $x = \cancel{1}.\ e$

$\frac{d^2y}{dx^2} = 2x - 2x\cdot\ln x - x$

$= x - 2x\cdot\ln x$

$\frac{d^2y}{dx^2} = 1-0>0$

$\therefore f(1) = 0$ 為極小值.

詳另頁!

7. $f(x) = x(x-1)^2$

$f'(x) = (x-1)^2 + 2x(x-1)$

$= 3x^2-4x+1.$

$= (x-1)(3x-1)$

$f''(x) = 6x-4$

4% (1) 臨界點 $(1.\ 0)(\frac{1}{3}.\ \frac{4}{27})$

4% (2) 反曲點 $(\frac{2}{3}.\ \frac{2}{27})$

4% (4) $x<\frac{2}{3}.\ f''(x)<0$: ∩

$x>\frac{2}{3}.\ f''(x)>0$: ∪

4% (3) $x<\frac{1}{3}.\ f'(x)>0$ ↑

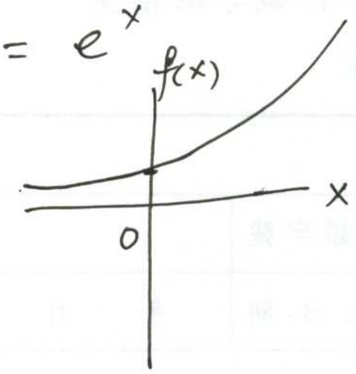
$\frac{1}{3}<x<1.\ f'(x)<0$ ↓

$x>1.\ f'(x)>0$ ↑

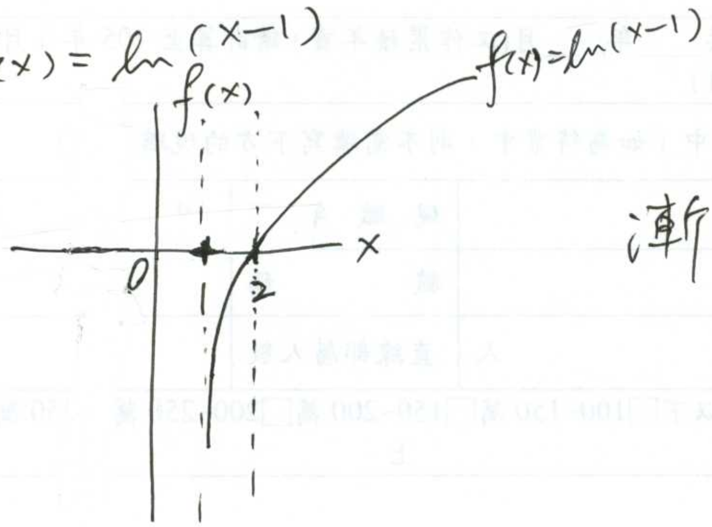
8

(1) $f(x)=\frac{2x^2}{x^2+3}=2-\frac{3}{x^2+3}$ 漸近線：$y=2$

(2) $f(x)=e^x$ 漸近線：$y=0$



(3) $f(x)=\ln(x-1)$



漸近線：$x=1$.

無水平漸近線

6. $y=\frac{\ln x}{x}=(x^{-1})(\ln x)$

$\frac{dy}{dx}=(-x^{-2})(\ln x)+x^{-2}$

$=\frac{1-\ln x}{x^2}$ 2%

$\frac{dy}{dx}=0.$ $\ln x=1.$ $x=e$ or $0.$ (不合) 3%

$\frac{d^2y}{dx^2}=(2x^{-3})(\ln x)+(-x^{-2})(x^{-1})-2x^{-3}$

$=\frac{2\ln x}{x^3}-\frac{1}{x^3}-\frac{2}{x^3}$

$=\frac{2(\ln x-\frac{3}{2})}{x^3}$

當 $x=e$ 時.

$\frac{d^2y}{dx^2}=\frac{2(1-\frac{3}{2})}{e^3}<0$ 2%

$\therefore f(e)=\frac{1}{e}$ 為極大值. 3%