<table>
<tr><td colspan="3">國立宜蘭大學 105 學年度第 2 學期 期中 考試試題紙</td><td>第 頁</td></tr>
<tr><td>考試科目</td><td>班級</td><td>學號</td><td>姓名</td></tr>
<tr><td>微積分二</td><td></td><td></td><td></td></tr>
</table>

1. Find the following integral. (48%)

(a) $\int \sin^3 x \cos^4 x dx$

(b) $\int \sec^3 x dx$

(c) $\int \frac{dx}{\sqrt{4x^2+1}}$

(d) $\int \frac{1}{x^2-5x+6} dx$

(e) $\int_0^1 \arcsin x dx$

(f) $\int_2^2 \frac{2x^3-4x-8}{x(x-1)(x^2+4)} dx$

2. Find the improper integral. (20%)

(a) $\int_{-1}^{2} \frac{1}{x^3} dx$

(b) $\int_0^{\infty} \frac{1}{\sqrt{x}(x+1)} dx$

3. Evaluate the following limits: (16%)

(a) $\lim_{x \to -\infty} \frac{x^2}{e^{-x}}$

(b) $\lim_{x \to \infty} \frac{\int_1^x \ln(e^{4t-1}) dt}{x}$

4. Find the sums of the following series. (16%)

(a) $\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$

(b) $\sum_{n=0}^{\infty} \frac{3}{2^n}$

1. (a) assume $u=\cos x$; $du=-\sin x\,dx$

$$\int \sin^3 x\cos^4 x\,dx=\int \sin^2 x\cos^4 x\sin x\,dx=\int (1-\cos^2 x)\cos^4 x\sin x\,dx$$

$$=\int (1-u^2)\cdot u^4\cdot(-du)=\int (u^6-u^4)\,du=\frac{1}{7}u^7-\frac{1}{5}u^5+C$$

$$=\frac{1}{7}\cos^7 x-\frac{1}{5}\cos^5 x+C$$

(b) $dv=\sec^2 x\,dx$, $v=\tan x$

$u=\sec x$, $du=\sec x\tan x$ $\quad \int u\,dv=uv-\int v\,du$

$$\int \sec^3 x\,dx=\sec x\tan x-\int \sec x\tan^2 x\,dx$$

$$=\sec x\tan x-\int \sec x(\sec^2 x-1)\,dx$$

$$=\sec x\tan x-\int \sec^3 x\,dx+\int \sec x\,dx$$

$$\Rightarrow 2\int \sec^3 x\,dx=\sec x\tan x+\int \sec x\,dx$$

$$\int \sec^3 x\,dx=\frac{1}{2}\sec x\tan x+\frac{1}{2}\ln|\sec x+\tan x|+C$$

(c) let $u=2x$, $a=1$, $2x=\tan\theta$, $\therefore dx=\frac{1}{2}\sec^2\theta\,d\theta$

$\sqrt{4x^2+1}=\sec\theta$

$$\int \frac{dx}{\sqrt{4x^2+1}}=\frac{1}{2}\int \frac{\sec^2\theta\,d\theta}{\sec\theta}=\frac{1}{2}\int \sec\theta\,d\theta=\frac{1}{2}\ln|\sec\theta+\tan\theta|+C$$

$$=\frac{1}{2}\ln\left|\sqrt{4x^2+1}+2x\right|+C$$

(d) $$\int \frac{1}{x^2-5x+6}\,dx=\int \frac{1}{(x-3)(x-2)}\,dx=\int \left(\frac{1}{x-3}-\frac{1}{x-2}\right)dx$$

$$=\ln|x-3|-\ln|x-2|+C$$

(e) let $dv=dx$, $v=x$

$u=\arcsin x$, $du=\frac{1}{\sqrt{1-x^2}}\,dx$

$$\int \arcsin x\,dx=x\cdot\arcsin x-\int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$=x\arcsin x+\sqrt{1-x^2}+C$$

$$\therefore \int_0^1 \arcsin x\,dx=\left[x\arcsin x+\sqrt{1-x^2}\right]_0^1=\frac{\pi}{2}-1$$

(f) $$\int_2^2 \frac{2x^3-4x-8}{x(x-1)(x^2+4)}\,dx=0$$

2.(a) $\int_{-1}^{2}\frac{dx}{x^3} = \int_{-1}^{0}\frac{dx}{x^3} + \int_{0}^{2}\frac{dx}{x^3}$

$\int_{-1}^{0}\frac{1}{x^3}dx = \lim_{a\to0^-}\left[-\frac{1}{2x^2}\right]_{-1}^{a} = \lim_{a\to0^-}\left(-\frac{1}{2a^2}+\frac{1}{2}\right) = -\infty$

或 $\int_{0}^{2}\frac{1}{x^3}dx = \lim_{b\to0^+}\left[-\frac{1}{2x^2}\right]_{b}^{2} = \lim_{b\to0^+}\left(-\frac{1}{8}+\frac{1}{2b^2}\right) = \infty$ $\therefore \int_{-1}^{2}\frac{dx}{x^3}$ divergent

(b) lot split the interval at $x=1$, $d\sqrt{x} = \frac{1}{2}\frac{1}{\sqrt{x}}dx \Rightarrow 2d\sqrt{x} = \frac{1}{\sqrt{x}}dx$

$\int_{0}^{\infty}\frac{1}{\sqrt{x}(x+1)}dx = \int_{0}^{1}\frac{dx}{\sqrt{x}(x+1)} + \int_{1}^{\infty}\frac{dx}{\sqrt{x}(x+1)} = \lim_{b\to0^+}\int_{b}^{1}\frac{2}{(\sqrt{x})^2+1^2}d\sqrt{x} + \lim_{c\to\infty}\int_{1}^{c}\frac{2}{(\sqrt{x})^2+1^2}d\sqrt{x}$

$= \lim_{b\to0^+}\left[2\arctan\sqrt{x}\right]_{b}^{1} + \lim_{c\to\infty}\left[2\arctan\sqrt{x}\right]_{1}^{c} = 2\left(\frac{\pi}{4}\right) - 0 + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right) = \pi$

3.(a) $\lim_{x\to-\infty}\frac{x^2}{e^{-x}} = \lim_{x\to-\infty}\frac{2x}{-e^{-x}} = \lim_{x\to-\infty}\frac{2}{e^{-x}} = 0$

(b) $\lim_{x\to\infty}\frac{\int_{1}^{x}\ln(e^{4t-1})dt}{x} = \lim_{x\to\infty}\frac{\int_{1}^{x}(4t-1)dt}{x} = \lim_{x\to\infty}\frac{\left[2t^2-t\right]_{1}^{x}}{x} = \lim_{x\to\infty}\frac{2x^2-x-1}{x}$

$= \lim_{x\to\infty}\left(2x-1-\frac{1}{x}\right) = \infty$ 　或原式 $= \lim_{x\to\infty}\frac{\int_{1}^{x}(4t-1)dt}{x} = \lim_{x\to\infty}\frac{4x-1}{1} = \infty$

4.(a) $a_n = \frac{1}{2n-1} - \frac{1}{2n+1}$

$S_n = \left(\frac{1}{1}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{2n-1}+\frac{1}{2n+1}\right) = 1 - \frac{1}{2n+1}$

$S = \lim_{n\to\infty}S_n = \lim_{n\to\infty}\left(1-\frac{1}{2n+1}\right) = 1$

(b) $a=3$, $r=\frac{1}{2}$

$S = \frac{a}{1-r} = \frac{3}{1-\frac{1}{2}} = 6$