Final Exam, Spring Semester 2017

for Freshman Classes of Food Science Department

10 Points for Each !

◎ Find the derivative of $f(x)$: i.e. $\dfrac{d f(x)}{dx}$

(1) $f(x) = \displaystyle\int_{3x^2}^{4x} t\sqrt{2-t^2}\,dt$

(2) $f(x) = \displaystyle\int_{\pi}^{x} \tan z \sec z\,dz$

◎ Find the following indefinite integrals:

(3) $\displaystyle\int x^2 \ln x\,dx$

(4) $\displaystyle\int \frac{e^{2x}}{e^x + 2}\,dx$

(5) $\displaystyle\int \frac{x\,dx}{x + 2 - \sqrt{x+2}}$

(6) $\displaystyle\int \frac{2x^4 - 3x^3 + 5x^2 - 2x + 2}{x^2 + 2}\,dx$

◎ Find the following definite integral

(7) $\displaystyle\int_0^{\pi/3} \sin^3 x\,dx$

(8) $\displaystyle\int_0^{\pi/3} e^x \sin 2x\,dx$

◎ Find the following improper integral

(9) $\displaystyle\int_0^{\infty} \frac{1}{(x+2)^4}\,dx$

(10) $\displaystyle\int_0^{\infty} e^{-2x}\,dx$

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(1) $16x\sqrt{2-16x^2} - 18x^3\sqrt{2-9x^4}$

(2) $\tan x \sec x$

(3) $\frac{x^3}{9}(3\ln x - 1) + C$

(4) $e^x - 2\ln(e^x+2) + C$

(5) $x + 2\sqrt{x+2} - 2\ln\left|\sqrt{x+2}-1\right| + C$ → (or $(\sqrt{x+2}+1)^2 - 2\ln\left|\sqrt{x+2}-1\right| + C$)

(6) $\frac{2}{3}x^3 - \frac{3}{2}x^2 + x + 2\ln(x^2+2) + C$

(7) $\frac{5}{24}$

(8) $\frac{1}{10}\left[4 + e^{\frac{\pi}{3}}(\sqrt{3}+2)\right]$

(9). $\frac{1}{24}$

(10). $\frac{1}{2}$

(1) Find $\dfrac{d}{dx} f(x)$

$$f(x) = \int_{3x^2}^{4x} t\sqrt{2-t^2}\,dt$$

Let $g(t) = t\sqrt{2-t^2}$

According to the fundamental theorem of calculus,

$$\frac{d}{dz}\int_a^z g(t)dt = g(z)$$

$$\because f(x) = \int_{3x^2}^{4x} g(t)dt = \int_{3x^2}^{a} g(t)dt + \int_a^{4x} g(t)dt$$

where a is an arbitrary number between $3x^2$ and $4x$

$$= \int_a^{4x} g(t)dt - \int_a^{3x^2} g(t)dt$$

$$\frac{df(x)}{dx} = \frac{d}{dx}\int_a^{4x} g(t)dt - \frac{d}{dx}\int_a^{3x^2} g(t)dt$$

Let $Z_1 = 4x$; $Z_2 = 3x^2$

$$\Rightarrow \frac{d}{dZ_1}\int_a^{Z_1} g(t)dt = g(Z_1) \quad ; \quad \frac{d}{dZ_2}\int_a^{Z_2} g(t)dt = Z_2$$

$$\frac{d}{d(4x)}\int_a^{4x} g(t)dt = g(4x) \quad ; \quad \frac{d}{d(3x^2)}\int_a^{3x^2} g(t)dt = g(3x^2)$$

$$\frac{1}{4}\cdot\frac{d}{dx}\int_a^{4x} g(t)dt = 4x\sqrt{2-(4x)^2} \quad ; \quad \frac{1}{6x}\cdot\frac{d}{dx}\int_a^{3x^2} g(t)dt = 3x^2\sqrt{2-(3x^2)^2}$$

$$\frac{d}{dx}\int_a^{4x} g(t)dt = 16x\sqrt{2-16x^2} \quad ; \quad \frac{d}{dx}\int_a^{3x^2} g(t)dt = 18x^3\sqrt{2-9x^4}$$

$$\therefore \frac{df(x)}{dx} = 16x\sqrt{2-16x^2} - 18x^3\sqrt{2-9x^4}$$

(2) Find $\dfrac{d}{dx} f(x)$

$$f(x) = \int_{\pi}^{x} \tan z \sec z\, dz$$

$$= \tan x \sec x$$

(3) $\int x^2 \ln x \, dx$

$= \frac{1}{3}\int (\ln x)\, d(x^3)$

$= \frac{1}{3}\left[x^3 \ln x - \int x^3 d(\ln x)\right]$

$= \frac{1}{3}\left[x^3 \ln x - \int x^3 \left(\frac{dx}{x}\right)\right]$

$= \frac{1}{3}\left(x^3 \ln x - \frac{x^3}{3}\right)$

$= \frac{x^3}{9}(3\ln x - 1) + C$

(4) $\int \frac{e^{2x}}{e^x+2}\, dx$

Let $u = e^x + 2$

$e^x = u-2$

$e^{2x} = (u-2)^2$

$du = e^x dx$

$dx = e^{-x} du$

$\therefore dx = \frac{du}{u-2}$

$\int \frac{e^{2x}}{e^x+2}\, dx$

$= \int \frac{(u-2)^2}{u} \cdot \frac{du}{u-2}$

$= \int \frac{(u-2)}{u}\, du$

$= \int du - 2\int \frac{du}{u}$

$= u - 2\ln|u| + C'$

$= e^x + 2 - 2\ln(e^x+2) + C'$

$= e^x - 2\ln(e^x+2) + C$

(5) $\int \frac{x\,dx}{x+2-\sqrt{x+2}}$

Let $u = \sqrt{x+2}$

$\Rightarrow u^2 = x+2$

or

$x = u^2 - 2$

$du = \frac{dx}{2\sqrt{x+2}}$

$dx = 2\sqrt{x+2}\, du$

$\Rightarrow dx = 2u\,du$

$\int \frac{x\,dx}{x+2-\sqrt{x+2}}$

$= \int \frac{u^2-2}{u^2-u} \cdot 2u\,du$

$= 2\int \frac{(u^2-2)du}{u-1}$

$= 2\int \frac{[(u+1)(u-1)-1]}{u-1}\, du$

$= 2\int (u+1)du - 2\int \frac{du}{u-1}$

$= (u+1)^2 - 2\ln|u-1| + C$

$= (\sqrt{x+2}+1)^2 - 2\ln\left|\sqrt{x+2}-1\right| + C$

or

$x + 2\sqrt{x+2} - 2\ln\left|\sqrt{x+2}-1\right| + C$

(6) $\int \frac{2x^4-3x^3+5x^2-2x+2}{x^2+2}\, dx$

Utilizing synthetic division

$$\begin{array}{ccccc|c} 2 & -3 & 5 & -2 & 2 & \\ & 0 & -4 & & & \\ & & 0 & 6 & & 0 \\ & & & 0 & -2 & -2 \\ \hline 2 & -3 & 1 & 4 & 0 & \end{array}$$

$\Rightarrow$ The original integral becomes

$\int (2x^2-3x+1)dx + 4\int \frac{x\,dx}{x^2+4}$

$= \frac{2}{3}x^3 - \frac{3}{2}x^2 + x + 2\ln(x^2+4) + C$

(7). $\int_0^{\frac{\pi}{3}} \sin^3 x\,dx$

$\because \int \sin^3 x\,dx$

$= -\int \sin^2 x\ (-\sin x\,dx)$

$= -\int (1-\cos^2 x)\,d(\cos x)$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

$\Rightarrow \int_0^{\frac{\pi}{3}} \sin^3 x\,dx$

$= -\cos x\Big|_0^{\frac{\pi}{3}} + \frac{1}{3}\cos^3 x\Big|_0^{\frac{\pi}{3}}$

$= -(\frac{1}{2}-1) + \frac{1}{3}[(\frac{1}{2})^3 - 1^3]$

$= -\frac{1}{2} + 1 + (-\frac{7}{24})$

$= \frac{5}{24}$

(8) $\int_0^{\frac{\pi}{3}} e^x \sin 2x\,dx$

Let

$I = \int e^x \sin 2x\,dx$

$= \int (\sin 2x)\,d(e^x)$

$= e^x \sin 2x - \int e^x (\cos 2x)\,2\cdot dx$

$= e^x \sin 2x - 2\int (\cos 2x)\,d(e^x)$

$= e^x \sin 2x - 2[e^x \cos 2x - \int e^x\cdot(-\sin 2x)\,2dx]$

$= e^x \sin 2x - 2e^x \cos 2x - 4\int e^x \sin 2x\,dx$

$\Rightarrow I = e^x \sin 2x - 2e^x \cos 2x - 4I$

$I = \frac{e^x(\sin 2x - 2\cos 2x)}{5} + C$

$\int_0^{\frac{\pi}{3}} e^x \sin 2x\,dx$

$= \frac{e^{\frac{\pi}{3}}(\sin\frac{2\pi}{3} - 2\cos\frac{2\pi}{3})}{5} - \frac{1\cdot[\sin(0) - 2\cos(0)]}{5}$

$= \frac{e^{\frac{\pi}{3}}(\frac{\sqrt{3}}{2}+1)}{5} + \frac{2}{5}$

(9) $\int_0^{\infty} \frac{1}{(x+2)^4}\,dx$

$\int \frac{dx}{(x+2)^4}$

$= \int \frac{d(x+2)}{(x+2)^4}$

$= -\frac{1}{3}\cdot\frac{1}{(x+2)^3} + C$

$\int_0^{\infty} \frac{1}{(x+2)^4}\,dx$

$= \lim_{b\to\infty}(-\frac{1}{3})[\frac{1}{(x+2)^3}]\Big|_0^b$

$= \frac{1}{24}$

(10). $\int_0^{\infty} e^{-2x}\,dx$

$\because \int e^{-2x}\,dx$

$= (-\frac{1}{2})\int e^{-2x}\,d(-2x)$

$= -\frac{1}{2}e^{-2x} + C$

$\int_0^{\infty} e^{-2x}\,dx$

$= \lim_{b\to\infty}(-\frac{1}{2})\,e^{-2x}\Big|_0^b$

$= \frac{-1}{2}(0-1) = \frac{1}{2}$

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