國立宜蘭大學 105 年度第一次微積分競試 解答

1(a) $\frac{3}{2}$	(b) 1	(c) -1	(d) $\frac{1}{2\sqrt{3}}$
(e) 4	(f) $\frac{1}{3}$	(g) $\frac{3}{2}$	(h) 1

2. $2e^{2x}(\tan 2x+\sec^2 2x)$	3. $\frac{(x-2)^2(2x^2+2x+3)}{(x^2+1)^{3/2}}$
4. $y=4x+25$	5. $\frac{2\pi}{3}+\sqrt{3}$
6. -2	7. 0
8. $1-\frac{x^2}{6}+\frac{x^4}{120}$	9. $y=5x+18,\quad x=3$
10. $-\frac{xe^{2x}}{2(2x+1)}+\frac{1}{4}e^{2x}+C$	11. $\frac{1}{1326}$
12. $\frac{1}{12}$	13. $(2, 4)$
14. $2\ln(2+\sqrt{3})$	15. $\frac{256\pi}{3}$
16. $-\frac{1}{3}\sqrt{16-x^2}(x^2+32)+C$	17. $\frac{a}{a^2+b^2}$
18(a). 10	18(b). $(-4,-27)$
19. $-\frac{1}{12\sqrt{3}}$	20. $\frac{1}{6}(e^9-1)$
21. $\frac{\pi}{8}(1-\cos 4)$	

1(a) $\lim_{x\to 0^+} 3x\csc 2x = \lim_{x\to 0^+}\frac{3x}{\sin 2x} = \lim_{x\to 0^+}\frac{3x}{\sin 2x}\cdot\frac{2}{2} = \frac{3}{2}$

(b) $\lim_{x\to -1}\frac{x^2+3x+2}{x+1} = \lim_{x\to -1}\frac{(x+1)(x+2)}{x+1} = \lim_{x\to -1}(x+2) = 1$

(c) $\lim_{x\to 0^+}\frac{\sin x(1-2\cos x)}{x} = \lim_{x\to 0^+}\frac{\sin x}{x}\cdot\lim_{x\to 0^+}(1-2\cos x) = -1$

(d) $\lim_{x\to 0^+}\frac{(\sqrt{x+3}-\sqrt{3})}{x} = \lim_{x\to 0^+}\frac{(\sqrt{x+3}-\sqrt{3})}{x}\frac{(\sqrt{x+3}+\sqrt{3})}{(\sqrt{x+3}+\sqrt{3})} = \lim_{x\to 0^+}\frac{x+3-3}{x(\sqrt{x+3}+\sqrt{3})} = \frac{1}{2\sqrt{3}}$

(e) $\lim_{x\to 0^+}\frac{(1-\cos 2x)^2}{x^4} = \lim_{x\to 0^+}\frac{4\sin^4 x}{x^4} = 4$

(f) $\lim_{x\to\infty}\frac{x^2+2x+3}{3x^2+1} = \lim_{x\to\infty}\frac{1+2/x+3/x^2}{3+1/x^2} = \frac{1}{3}$

(g) $\lim_{x\to 0^+}\frac{\sin 3x}{\sin 2x} = \lim_{x\to 0^+}\frac{2x}{3x}\cdot\frac{3}{2}\cdot\frac{\sin 3x}{\sin 2x} = \frac{3}{2}$

(h) $\lim_{x\to 3}\frac{\sqrt{x+6}}{3} = 1$

2. $y = e^{2x}\tan 2x$, $\frac{dy}{dx} = 2e^{2x}\tan 2x + 2e^{2x}\sec^2 2x = 2e^{2x}(\tan 2x + \sec^2 2x)$

3. $y = \frac{(x-2)^3}{\sqrt{x^2+1}}$, $\frac{dy}{dx} = \frac{3(x-2)^2\sqrt{x^2+1} - (x-2)^3\cdot\frac{1}{2}\cdot\frac{2x}{\sqrt{x^2+1}}}{(x^2+1)} = \frac{(x-2)^2(3x^2+3-x^2+2x)}{(x^2+1)^{3/2}}$

$= \frac{(x-2)^2(2x^2+2x+3)}{(x^2+1)^{3/2}}$

4. $y = \frac{x}{x+4}$, $\frac{dy}{dx} = \frac{x+4-x}{(x+4)^2} = \frac{4}{(x+4)^2}$，令切點為 (x_0, y_0)，$y_0 = \frac{x_0}{x_0+4}$，$m = \frac{4}{(x_0+4)^2}$

切線 $y - \frac{x_0}{x_0+4} = \frac{4}{(x_0+4)^2}(x-x_0)$ 通過 $(-4,9)$，得 $x_0 = -5,\quad y_0 = 5$，$m = 4 \quad\Rightarrow$

$y-5 = 4(x+5) \quad\Rightarrow\quad y = 4x+25$

5. $f(x) = x + 2\sin x$，$f' = 1+2\cos x = 0$，$\cos x = -\frac{1}{2}$，$x = \frac{2\pi}{3}$，$f(\frac{2\pi}{3}) = \frac{2\pi}{3}+\sqrt{3}$

$f(0) = 0$，$f(\pi) = \pi$，$\frac{2\pi}{3}+\sqrt{3} > \pi$

6. $f(x) = \frac{1}{x}+x$，$f'(x) = -\frac{1}{x^2}+1 = \frac{x^2-1}{x^2} = 0$，$x = \pm 1$，$f(1) = 2,\quad f(-1) = -2$

$f''(x) = \frac{2}{x^3}$，$x = 1,\quad f''(1) > 0$ 有相對最小值，$x = -1,\quad f''(-1) < 0$ 有相對最大值

7. $(x^2+y^2)^2 = 4x^2y$，$2(x^2+y^2)(2x+2y\frac{dy}{dx}) = 8xy + 4x^2\frac{dy}{dx}$，

at $(1,1) \Rightarrow 8+8\frac{dy}{dx} = 8+4\frac{dy}{dx}, \Rightarrow \frac{dy}{dx} = 0$

8. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$， $\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots$

9. $y = \frac{5x^2 + 3x + 2}{x-3} = 5x + 18 + \frac{56}{x-3}$， $y = 5x + 18$ 為斜漸近線，$x = 3$ 為垂直漸近線

10. 令 $u = xe^{2x}, dv = \frac{dx}{(2x+1)^2}$， $du = (e^{2x} + 2xe^{2x})dx, v = -\frac{1}{2(2x+1)}$

$$\int \frac{xe^{2x}}{(2x+1)^2} dx = -\frac{1}{2}\frac{xe^{2x}}{(2x+1)} + \frac{1}{2}\int \frac{e^{2x}(2x+1)}{(2x+1)} dx = -\frac{1}{2}\frac{xe^{2x}}{(2x+1)} + \frac{1}{4}e^{2x} + C$$

11. 令 $u = 1 - \sqrt{x}, u - 1 = -\sqrt{x}$, $(u-1)^2 = x$, $2(u-1)du = dx$, $x = 0, u = 1$, $x = 1, u = 0$

$$\int_0^1 (1-\sqrt{x})^{50} dx = \int_1^0 (u)^{50} 2(u-1)du = 2\int_1^0 (u^{51} - u^{50)}du = 2\int_0^1 (u^{50} - u^{51)}du$$

$$= 2[\frac{1}{51}u^{51} - \frac{1}{52}u^{52}]_0^1 = 2(\frac{1}{51} - \frac{1}{52}) = \frac{1}{1326}$$

12. $\int_1^b \frac{1}{(3x+1)^2} dx = \frac{1}{3}\int_1^b \frac{1}{(3x+1)^2} d(3x+1) = -\frac{1}{3}\left[\frac{1}{3x+1}\right]_1^b = -\frac{1}{3}[\frac{1}{3b+1} - \frac{1}{4}]$

$$\int_1^\infty \frac{1}{(3x+1)^2} dx = \lim_{b\to\infty}\int_1^b \frac{1}{(3x+1)^2} dx = -\frac{1}{3}\lim_{b\to\infty}[\frac{1}{3b+1} - \frac{1}{4}] = \frac{1}{12}$$

13. 令 $x = t, y = t^2$, $R = (t-6)^2 + (t^2-3)^2$, $\frac{dR}{dt} = 0 \Rightarrow 2t^3 - 5t - 6 = 0, t = 2, \Rightarrow (2,4)$

14. $y = \ln(\cos x), \frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$, $ds = \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \sec x dx$

$$L = \int_{-\pi/3}^{\pi/3} \sec x dx = \left[\ln(\sec x + \tan x)\right]_{-\pi/3}^{\pi/3} = \ln(2+\sqrt{3}) - \ln(2-\sqrt{3}) = \ln(\frac{2+\sqrt{3}}{2-\sqrt{3}})$$

$$= \ln(\frac{(2+\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}) = \ln(2+\sqrt{3})^2 = 2\ln(2+\sqrt{3})$$

15. $y = 8x - 2x^2, y = 4x - x^2$, $4x - x^2 = 8x - 2x^2 \Rightarrow x^2 - 4x = 0$, $x = 0, x = 4$ (交點)

$$V = 2\pi\int_0^4 (x+2)(8x - 2x^2 - 4x + x^2)dx = 2\pi\int_0^4 (8x + 2x^2 - x^3)dx = \frac{256\pi}{3}$$

16. 令 $x = 4\sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, dx = 4\cos\theta\, d\theta$， $\sqrt{16 - x^2} = \sqrt{16 - 16\sin^2\theta} = 4\cos\theta$

$$\int \frac{x^3}{\sqrt{16-x^2}} dx = \int \frac{64\sin^3\theta 4\cos\theta\, d\theta}{4\cos\theta} = 64\int \sin^3\theta\, d\theta = 64\int (1-\cos^2\theta)\sin\theta\, d\theta$$

令 $u = \cos\theta, du = -\sin\theta\, d\theta$， $u = \cos\theta = \frac{\sqrt{16-x^2}}{4}$

原式$= 64\int (u^2 - 1)du = \frac{64}{3}u^3 - 64u + C = \frac{(16-x^2)^{3/2}}{3} - 16(16-x^2)^{1/2} + C$

$$=\frac{1}{3}\sqrt{16-x^2}(16-x^2-48)+C=-\frac{1}{3}\sqrt{16-x^2}(x^2+32)+C$$

17. $\vec{r}(t)=a\cos t\,\vec{i}+a\sin t\,\vec{j}+bt\,\vec{k},\quad \vec{v}(t)=\frac{d\vec{r}}{dt}=-a\sin t\,\vec{i}+a\cos t\,\vec{j}+b\,\vec{k},$

$$\vec{T}=\frac{\vec{v}}{|\vec{v}|}=\frac{1}{\sqrt{a^2+b^2}}(-a\sin t\,\vec{i}+a\cos t\,\vec{j}+b\,\vec{k}),$$

$$\kappa=\frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|=\frac{1}{\sqrt{a^2+b^2}}\left|\frac{1}{\sqrt{a^2+b^2}}(-a\cos t\,\vec{i}-a\sin t\,\vec{j})\right|=\frac{a}{a^2+b^2}$$

18. (a) $t=3\Rightarrow x=2,\ y=3$

$$\frac{dT}{dt}=\frac{\partial T}{\partial x}\frac{dx}{dt}+\frac{\partial T}{\partial y}\frac{dy}{dt}=(2x)\left[\frac{1}{2}(1+t)^{-1/2}\right]+(3y^2)(\frac{1}{3})=4\cdot\frac{1}{4}+27\cdot\frac{1}{3}=10$$

(b) $\nabla T=(T_x,T_y)=(4,27)\quad\Rightarrow\quad -\nabla T=(-4,-27)$

19. 令 $x=\overline{AB},\ y=\overline{AC},\ \theta=\angle A,\quad A=\frac{1}{2}xy\sin\theta,\quad \frac{dA}{dt}=\frac{1}{2}(y\sin\theta\frac{dx}{dt}+x\sin\theta\frac{dy}{dt}+xy\cos\theta\frac{d\theta}{dt}$

$$\frac{dA}{dt}=0\quad\Rightarrow\quad 0=\frac{1}{2}\left[30\sin 30^\circ\times 3+20\sin 30^\circ\times(-2)+20\times 30\cos 30^\circ\frac{d\theta}{dt}\right],\quad \frac{d\theta}{dt}=-\frac{1}{12\sqrt{3}}$$

20. $R:0\le y\le 1,3y\le x\le 3\quad\Rightarrow\quad R:0\le x\le 3,0\le y\le\frac{x}{3}$

$$\int_0^1\int_{3y}^3 e^{x^2}dxdy=\int_0^3\int_0^{\frac{x}{3}}e^{x^2}dydx=\int_0^3 ye^{x^2}\Big|_0^{\frac{x}{3}}dx=\frac{1}{3}\int_0^3 xe^{x^2}dx=\frac{1}{6}(e^9-1)$$

21. $R:0\le\theta\le\frac{\pi}{4},\quad 0\le r\le 2$

$$\iint_R\sin(x^2+y^2)dA=\int_0^{\frac{\pi}{4}}\int_0^2\sin(r^2)rdrd\theta=\frac{1}{2}\int_0^{\frac{\pi}{4}}\int_0^4\sin(u)dud\theta=\frac{\pi}{8}(1-\cos 4)$$