國立宜蘭大學 105 年度第二次微積分競試 解答

班級： 學號： 姓名：

1. 30	2. 0
3. 1	4. $\frac{1}{3}$
5. $\frac{1}{2}$	6. 1
7. $(1,0)$	8. $(60,-148)$
9. $y=-1,\quad x=\pm 2$	10. 2
11. $-\tan t$ or $-\frac{x}{y}$	12. $\frac{1}{101}$
13. $3(x^2-6x+18)e^{\frac{x}{3}}+C$	14. $-\frac{1}{4(2x+1)^2}+C$
15. 1	16. $\frac{256}{3}$
17. $\ln(2+\sqrt{3})$	18. $-\frac{xz^2\cos xyz}{\sin xyz+xyz\cos xyz}$
19. 6	20. e^4-1
21. $\frac{625}{24}$	22. 6π
23. $-\frac{1}{4}$	24. $x-\frac{x^3}{3}+\frac{x^5}{5}$

1. Let $y = f(x) = \left(\dfrac{x+2}{4}\right)^{1/3}$, $y^3 = \dfrac{x+2}{4}$, $x = 4y^3 - 2$, $\Rightarrow$ $g(x) = 4x^3 - 2$, $g(2) = 30$

2. $\lim\limits_{x\to 0} \dfrac{\tan^2 x}{x} = \lim\limits_{x\to 0} (\dfrac{\sin x}{\cos x})^2 \cdot \dfrac{1}{x} = \lim\limits_{x\to 0} (\dfrac{\sin x}{x}) \cdot \lim\limits_{x\to 0} (\sin x) \cdot \lim\limits_{x\to 0} (\dfrac{1}{\cos x})^2 = 1 \cdot 0 \cdot 1 = 0$

3. $\lim\limits_{\theta\to\pi/2} \dfrac{\cos\theta}{\cot\theta} = \lim\limits_{\theta\to\pi/2} \dfrac{\cos\theta}{\cos\theta/\sin\theta} = \lim\limits_{\theta\to\pi/2} \sin\theta = 1$

4. $\because$ $x - 1 = (\sqrt[3]{x} - 1)(\sqrt[3]{x^2} + \sqrt[3]{x} + 1)$

$$\lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{x - 1} = \lim_{x\to 1} (\frac{\sqrt[3]{x} - 1}{(\sqrt[3]{x} - 1)(\sqrt[3]{x^2} + \sqrt[3]{x} + 1)} = \lim_{x\to 1} \frac{1}{\sqrt[3]{x^2} + \sqrt[3]{x} + 1} = \frac{1}{3}$$

5. $\lim\limits_{x\to 1} \dfrac{1 + x(\ln x - 1)}{(x-1)\ln x}$ $\dfrac{0}{0}$ type, L'Hôpital's rule can be applied

$$= \lim_{x\to 1} \frac{(\ln x - 1) + x \cdot \frac{1}{x}}{\ln x + \frac{x-1}{x}} = \lim_{x\to 1} \frac{x \ln x}{x \ln x + x - 1}$$ $\dfrac{0}{0}$ type again

$$= \lim_{x\to 1} \frac{\ln x + x \cdot \frac{1}{x}}{\ln x + x \cdot \frac{1}{x} + 1} = \lim_{x\to 1} \frac{\ln x + 1}{\ln x + 2} = \frac{1}{2}$$

6. Let $y = [\cos(\dfrac{\pi}{2} - x)]^x$, $\ln y = x \ln[\cos(\dfrac{\pi}{2} - x)] = \dfrac{\ln[\cos(\frac{\pi}{2} - x)]}{1/x}$

$\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \dfrac{\ln[\cos(\frac{\pi}{2} - x)]}{1/x}$ $\dfrac{-\infty}{\infty}$ type, L'Hôpital's rule can be applied

$$= \lim_{x\to 0^+} \frac{\frac{1}{\cos(\pi/2 - x)}[-\sin(\pi/2 - x)](-1)}{-1/x^2} = \lim_{x\to 0^+} \frac{-x^2 \sin(\pi/2 - x)}{\cos(\pi/2 - x)} = \lim_{x\to 0^+} \frac{-x^2}{\cos(\pi/2 - x)} \cdot \lim_{x\to 0^+} \sin(\pi/2 - x)$$

$\because$ $\lim\limits_{x\to 0^+} \dfrac{-x^2}{\cos(\pi/2 - x)} = \lim\limits_{x\to 0^+} \dfrac{-2x}{-\sin(\pi/2 - x)(-1)} = 0$ and $\lim\limits_{x\to 0^+} \sin(\pi/2 - x) = 1$

$\therefore$ $\lim\limits_{x\to 0^+} \ln y = 0$ $\Rightarrow$ $\lim\limits_{x\to 0^+} y = e^0 = 1$, $\lim\limits_{x\to 0^+} [\cos(\dfrac{\pi}{2} - x)]^x = 1$

7. $f(x) = (x-1)^{\frac{5}{3}}$, $f'(x) = \dfrac{5}{3}(x-1)^{\frac{2}{3}}$, $f''(x) = \dfrac{10}{9}(x-1)^{-\frac{1}{3}}$

$x = 1$時 f''不存在 且 $f'(1) = 0$(切線存在) $\Rightarrow$ $(1, 0)$為反曲點

8. $f(x) = x^3 - 3x^2 - 24x + 32$

$f'(x)=3x^2-6x-24=3(x^2-2x-8)=3(x-4)(x+2)$， $f'(x)=0 \Rightarrow x=-2,\quad x=4$

$f(-6)=-216-108+144+32=-148$，
$f(-2)=-8-12+48+32=60$
$f(4)=64-48-96+32=-48$， (最大值，最小值) $=(60,-148)$

9. $f(x)=\dfrac{x^2}{4-x^2}$， $\lim\limits_{x\to\infty}\dfrac{x^2}{4-x^2}=\lim\limits_{x\to\infty}\dfrac{1}{\frac{4}{x^2}-1}=-1$， $\lim\limits_{x\to-\infty}\dfrac{x^2}{4-x^2}=-1$

水平漸近線 $y=-1$，垂直漸近線 $x=\pm 2$

10. $(4-x)y^2=3x^2$，等號兩邊對 x 微分 $\Rightarrow\ -y^2+(4-x)\cdot 2y\cdot\dfrac{dy}{dx}=6x$

$x=2, y=2$， $\Rightarrow\quad -4+(4-2)\cdot 4\cdot\dfrac{dy}{dx}=12$， $m=\dfrac{dy}{dx}=\dfrac{16}{8}=2$

11. $x=\sin t,\quad y=\cos t,\quad \Rightarrow\quad dx=\cos t dt,\quad dy=-\sin t dt,\quad \Rightarrow \dfrac{dy}{dx}=\dfrac{-\sin t}{\cos t}=-\tan t=-\dfrac{x}{y}$

12. $\int_0^{\pi/2}\sin^{100}(x)\cos(x)dx=\int_0^{\pi/2}\sin^{100}(x)d\sin(x)=\dfrac{1}{101}\sin^{101}(x)\Big|_0^{\pi/2}=\dfrac{1}{101}$

13. $\int x^2e^{x/3}dx$ $\quad\int udv=uv-\int vdu$ 令 $u=x^2,\quad dv=e^{x/3}dx,\quad\Rightarrow\quad du=2xdx,\quad v=3e^{x/3}$

$=3x^2e^{x/3}-6\int xe^{x/3}dx$ $\quad$ 令 $u=x,\quad dv=e^{x/3}dx,\quad\Rightarrow\quad du=dx,\quad v=3e^{x/3}$

$=3x^2e^{x/3}-6[3xe^{x/3}-3\int e^{x/3}dx]=3x^2e^{x/3}-18xe^{x/3}+18\int e^{x/3}dx$

$=3x^2e^{x/3}-18xe^{x/3}+54e^{x/3}+C==3(x^2-6x+18)e^{x/3}+C$

14. $\int\dfrac{1}{(2x+1)^3}dx$ $\quad$ 令 $z=2x+1.\quad dz=2dx$

$=\int\dfrac{1}{z^3}\cdot\dfrac{dz}{2}=\dfrac{1}{2}\int z^{-3}dz=\dfrac{1}{2}\cdot\dfrac{1}{-2}z^{-2}+C=-\dfrac{1}{4(2x+1)^2}+C$

15. $f(x,y)=x^2-y^2,\quad 2y-x^2=0,\qquad x^2=2y$ 代入

$f(y)=2y-y^2=-(y^2-2y+1)+1=-(y-1)^2+1$， 最大值為 1

16. $y=x^2,\quad y=4x+12,\quad\Rightarrow\quad x^2=4x+12,\quad (x-6)(x+2)=0,\quad x=-2,\quad x=6$

$A=\int_{-2}^{6}(4x+12-x^2)dx=\int_{-2}^{6}(-x^2+4x+12)dx=\left[-\dfrac{1}{3}x^3+2x^2+12x\right]_{-2}^{6}$

$=(-72+72+72)-(\dfrac{8}{3}+8-24)=\dfrac{256}{3}$

17. $y=\ln(\cos x),\quad x\in[0,\dfrac{\pi}{3}],\qquad \dfrac{dy}{dx}=\dfrac{-\sin x}{\cos x}=-\tan x$

$$ds = \sqrt{1+(\frac{dy}{dx})^2}\ dx = \sqrt{1+\tan^2 x}\ dx = \sec x dx$$

$$L = \int ds = \int_0^{\pi/3} \sec x dx = [\ln(\sec x + \tan x)]_0^{\pi/3} = \ln(2+\sqrt{3}) - \ln(1) = \ln(2+\sqrt{3})$$

18. 令 $f(x,y,z) = x^9 + z\sin xyz = 0$，則 $\frac{\partial f}{\partial z} = \sin xyz + xyz\cos xyz$

$$\frac{\partial f}{\partial x} = 9x^8 + yz^2\cos xyz, \quad \frac{\partial f}{\partial y} = xz^2\cos xyz$$

故 $\underline{\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{9x^8 + yz^2\cos xyz}{\sin xyz + xyz\cos xyz}}$ $\underline{\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{xz^2\cos xyz}{\sin xyz + xyz\cos xyz}}$

19. Suppose that $D_{\mathbf{u}} f(3,15) = -10$ and $D_{\mathbf{v}} f(3,15) = 15$, where $\mathbf{u} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$ and $\mathbf{v} = \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$. Find $f_x(3,15)$.

因 $\nabla f = f_x\hat{i} + f_y\hat{j}$ 得 $D_{\mathrm{u}} f = \nabla f \bullet \mathbf{u} = \frac{3}{5} f_x - \frac{4}{5} f_y = -10$ ....(1)

$$D_{\mathrm{v}} f = \nabla f \bullet \mathbf{v} = \frac{4}{5} f_x + \frac{3}{5} f_y = 15 \text{ ....(2)}$$

(1)x3+(2)x4 $5f_x = 30$ $\underline{f_x(3,15) = 6}$

20. $\int_0^4 \int_{y/2}^2 e^{x^2} dxdy = \int_0^2 \int_0^{2x} e^{x^2} dydx = \int_0^2 \left(ye^{x^2} \right]_0^{2x} \Big) dx$

$$= \int_0^2 2xe^{x^2} dx = e^{x^2} \Big]_0^2 = \underline{e^4 - 1}$$

21. $\iint_R xydA = \int_0^5 \int_0^{5-y} xydxdy = \int_0^5 \left. \frac{x^2 y}{2} \right]_0^{5-y} dy = \int_0^5 \left(\frac{25y}{2} - 5y^2 + \frac{y^3}{2} \right) dy$

$$= \left[\frac{25y^2}{4} - \frac{5y^3}{3} + \frac{y^4}{8} \right]_0^5 = \underline{\frac{625}{24}}$$

22. $\iint_R (x^2 + y)dA$ $\because x = r\cos\theta,\quad y = r\sin\theta,\quad dA = rd\theta dr$

$$= \int_1^{\sqrt{5}} \int_0^{2\pi} (r^2\cos^2\theta + r\sin\theta) rd\theta dr = \int_1^{\sqrt{5}} \int_0^{2\pi} (r^3\cos^2\theta + r^2\sin\theta) d\theta dr$$

$$= \int_1^{\sqrt{5}} \int_0^{2\pi} r^3 \frac{1+\cos 2\theta}{2} + r^2 \sin\theta) d\theta dr = \int_1^{\sqrt{5}} \int_0^{2\pi} (\frac{r^3}{2} + \frac{r^3 \cos 2\theta}{2} + r^2 \sin\theta) d\theta dr$$

$$= \int_1^{\sqrt{5}} (\pi r^3) dr \qquad \because \int_0^{2\pi} \cos 2\theta\, d\theta = 0, \quad \int_0^{2\pi} \sin\theta\, d\theta = 0,$$

$$= \frac{\pi}{4} [r^4]_1^{\sqrt{5}} = \frac{\pi}{4} (25-1) = 6\pi$$

23. Find the absolute minimum of the function $f(x,y) = x^2 + 4y^2 - x$ on the indicated closed and bounded set *R*. *R* is the disk $x^2 + y^2 \le 9$.

Find the critical points of f. $\nabla f = \vec{0}$

由 $f_x = 0$ 得 $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

由 $f_y = 0$ 得 $8y = 0 \Rightarrow y = 0$

There is a critical point at $(\frac{1}{2}, 0)$, which is in the domain.

Check the endpoints, $x^2 + y^2 \le 9 \Rightarrow -3 \le x \le 3$, 將 $y^2 = 9 - x^2$ 代入原式

$$f(x,y) = x^2 + 4(9 - x^2) - x = -3x^2 - x + 36$$

微分 $f'(x,y) = -6x - 1 = 0 \Rightarrow x = -\frac{1}{6}$

There are critical endpoints at $(-\frac{1}{6}, \frac{\sqrt{323}}{6}), (-\frac{1}{6}, -\frac{\sqrt{323}}{6})$.

再加上考慮兩端點 $(-3,0), (3,0)$

(x,y)	$(-3,0)$	$(-\frac{1}{6}, \frac{\sqrt{323}}{6})$	$(-\frac{1}{6}, -\frac{\sqrt{323}}{6})$	$(\frac{1}{2}, 0)$	$(3,0)$
$f(x,y)$	12	$\frac{433}{12}$	$\frac{433}{12}$	$-\frac{1}{4}$	6

absolute maximum $\underline{\frac{433}{12}}$, absolute minimum $\underline{-\frac{1}{4}}$

24. 無窮等比級數 $\frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots,\quad -1<x<1$

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=1-x^2+x^4-x^6+\cdots,$$

$$\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}\quad\Rightarrow\quad \tan^{-1}x+C=\int\frac{1}{1+x^2}dx=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots$$

$\because\quad \tan^{-1}(0)=0\quad\Rightarrow\quad C=0,\qquad \therefore\quad \tan^{-1}x\approx x-\frac{1}{3}x^3+\frac{1}{5}x^5$ (取前三項)