國立宜蘭大學 106 年度第一次微積分競試 解答

1. $\frac{1}{2\sqrt{2}}$	2. $-\frac{2}{x^3}$
3. $\frac{2}{3}$	4. $\frac{1}{6}$
5. $y=\frac{x}{12}+\frac{4}{3}$	6. 最大值17，最小值1
7. $\frac{5}{2}$	8. $\frac{21}{5\sqrt{6}}$
9. $-\frac{1}{2}<x<\frac{1}{2}$	10. $-\frac{3}{2}$
11. $x=-\sqrt{3},\quad x=\sqrt{3},\quad y=-3$	12. $(\frac{\sqrt{6}}{3},2),\quad(-\frac{\sqrt{6}}{3},2)$
13. $\frac{20}{3}$	14. $\frac{1165}{4}$
15. 7π	16. $3(\sqrt[3]{2}+1)$
17. (a)收斂 (b)收斂	18. $-1\le x<1$
19. 120	20. $\frac{1-y\cos(xy+z)}{\cos(xy+z)}$
21a. $\frac{23}{27}$	21b. $\frac{3}{2\sqrt{2}}-1$
22. 1	23. $\frac{52}{9}$
24. $\frac{65\pi}{8}$	

1. $\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}=\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}\cdot\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}=\lim_{x\to 0}\frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})}=\lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}}$

$=\frac{1}{2\sqrt{2}}$

2. $\lim_{\Delta x\to 0}\frac{\frac{1}{(x+\Delta x)^2}-\frac{1}{x^2}}{\Delta x}=\frac{d}{dx}(\frac{1}{x^2})=-\frac{2}{x^3}$

3. $f(x)=\sqrt{x}$, $\lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}f(\frac{i}{n})\right]=\int_0^1\sqrt{x}dx=\frac{2}{3}[x^{\frac{3}{2}}]_0^1=\frac{2}{3}$

4. $f(x)=x^2,\quad x>0,\quad\Rightarrow\quad f^{-1}(x)=\sqrt{x}$, $g(x)=\frac{d}{dx}[f^{-1}(x)]=\frac{1}{f'(f^{-1}(x))}=\frac{1}{2\sqrt{x}}$,

$g(9)=\frac{1}{6}$

5. $f(x)=\sqrt[3]{x}=x^{1/3}$, $\frac{dy}{dx}=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}}$, at (8,2), $m=\frac{dy}{dx}=\frac{1}{3}\cdot\frac{1}{4}=\frac{1}{12}$

Tangent line $y-2=\frac{1}{12}(x-8)\quad\Rightarrow\quad y=\frac{1}{12}x+\frac{4}{3}$

6. $f(x)=x^3-x^2-x+2$, $f'(x)=3x^2-2x-1=(3x+1)(x-1)$, $-\frac{1}{3}$ is excluded from $[0,3]$

$f(1)=1-1-1+2=1$, $f(0)=2$, $f(3)=27-9-3+2=17$ $\therefore$ 最大值 17，最小值 1

7. $\int_0^2|2x-1|dx=\int_0^{1/2}(1-2x)dx+\int_{1/2}^2(2x-1)dx=[x-x^2]_0^{1/2}+[x^2-x]_{1/2}^2=\frac{1}{4}+2-(\frac{1}{4}-\frac{1}{2})=\frac{5}{2}$

8. A point $P(0,0,0)$ on the plane $-2x+y+z=0$,

the distance between the point $P(0,0,0)$ and the plane $10x-5y-5z-21=0$ is

$D=\frac{|0-0-0-21|}{\sqrt{100+25+25}}=\frac{21}{\sqrt{150}}=\frac{21}{5\sqrt{6}}$

9. Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{n+1}{n+2}(-2x)^n}{\frac{n}{n+1}(-2x)^{n-1}}\right|=\lim_{n\to\infty}\frac{(n+1)^2}{n(n+2)}|2x|\to|2x|$, $|2x|<1\Rightarrow-\frac{1}{2}<x<\frac{1}{2}$

$x=\pm\frac{1}{2}$, $\lim_{n\to\infty}a_n\neq 0$, the series diverge , $\therefore$ the interval of convergence is $-\frac{1}{2}<x<\frac{1}{2}$

10. $f(x^5)=3f(x)-4x^2+5x-6$, let $g(x)=x^5$, $f(g(x))=3f(x)-4x^2+5x-6$

$$\frac{d}{dx}[f(g(x))]=\frac{d}{dx}[3f(x)-4x^2+5x-6],\quad \frac{df(g)}{dg}\frac{dg}{dx}=3f'(x)-8x+5$$

$f'(x^5)\cdot 5x^4=3f'(x)-8x+5$, at $x=1 \Rightarrow 5f'(1)=3f'(1)-8+5 \Rightarrow f'(1)=-3/2$

11. $f(x)=\dfrac{3x^2+5}{3-x^2}=-3-\dfrac{14}{x^2-3}=-3-\dfrac{14}{(x+\sqrt{3})(x-\sqrt{3})}$

Asymptotes : $x=\sqrt{3},\quad x=-\sqrt{3},\quad y=-3$

12. $f(x)=\log_2(3x^2+2)=\dfrac{\ln(3x^2+2)}{\ln 2}$, $f'(x)=\dfrac{1}{\ln 2}\cdot\dfrac{6x}{3x^2+2}$,

$$f''(x)=\frac{1}{\ln 2}\cdot\frac{6(3x^2+2)-6x(6x)}{(3x^2+2)^2}=\frac{1}{\ln 2}\cdot\frac{18x^2+12-36x^2}{(3x^2+2)^2}=\frac{1}{\ln 2}\cdot\frac{-18x^2+12}{(3x^2+2)^2}$$

$$f''(x)=0 \Rightarrow 18x^2-12=0 \Rightarrow 3x^2-2=0 \Rightarrow x=\pm\sqrt{\frac{2}{3}}=\pm\frac{\sqrt{6}}{3}$$

$$f\left(\frac{\sqrt{6}}{3}\right)=\log_2\left(3\cdot\frac{6}{9}+2\right)=2,\quad f\left(-\frac{\sqrt{6}}{3}\right)=\log_2\left(3\cdot\frac{6}{9}+2\right)=2$$

There are two inflection points $\left(\dfrac{\sqrt{6}}{3},2\right)$ and $\left(-\dfrac{\sqrt{6}}{3},2\right)$

13. $\displaystyle\int_0^2\int_{-1}^1(x^2y+y^2)\,dxdy=\int_0^2\left[\frac{1}{3}x^3y+y^2x\right]_{x=-1}^{x=1}dy=\int_0^2\left(\frac{2}{3}y+2y^2\right)dy=\left[\frac{1}{3}y^2+\frac{2}{3}y^3\right]_0^2=\frac{20}{3}$

14. mean value = $\displaystyle\frac{1}{3}\int_0^3(t^3-7t^2+17t+280)\,dt=\frac{1}{3}\left[\frac{1}{4}t^4-\frac{7}{3}t^3+\frac{17}{2}t^2+280t\right]_0^3$

$$=\frac{1}{3}\left[\frac{81}{4}-63+\frac{153}{2}+840\right]=\frac{1}{3}\cdot\frac{(81-252+306+3360)}{4}=\frac{1}{3}\cdot\frac{3495}{4}=\frac{1165}{4}$$

15. $x=\cos 7t,\quad y=\sin 7t \Rightarrow \dfrac{dx}{dt}=-7\sin 7t,\quad \dfrac{dy}{dt}=7\cos 7t$

$$s=\int_0^\pi\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^\pi\sqrt{(-7\sin 7t)^2+(7\cos 7t)^2}\,dt=\int_0^\pi\sqrt{49(\sin^2 7t+\cos^2 7t)}\,dt$$

$$=\int_0^\pi 7\,dt=7t\Big|_0^\pi=\underline{7\pi}$$

16. $\displaystyle\int_0^3\frac{dx}{(x-1)^{2/3}}=\int_0^1\frac{dx}{(x-1)^{2/3}}+\int_1^3\frac{dx}{(x-1)^{2/3}}$

$$\int_0^1\frac{dx}{(x-1)^{2/3}}=\lim_{b\to1^-}\int_0^b\frac{dx}{(x-1)^{2/3}}=\lim_{b\to1^-}3[(x-1)^{1/3}]_0^b=3$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{a\to 1^+} \int_a^3 \frac{dx}{(x-1)^{2/3}} = \lim_{a\to 1^+} 3[(x-1)^{1/3}]_a^3 = 3\sqrt[3]{2}$$

$$\therefore \int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2} = 3(\sqrt[3]{2}+1)$$

17. (a) let $a_n = \dfrac{6n^2-6n+5}{12n^7+3n-8}$, $b_n = \dfrac{1}{n^5}$; and $\sum_{n=1}^{\infty} \dfrac{1}{n^5}$ converges (p-series with p = 5 >1)

Limit comparison test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{6n^2-6n+5}{12n^7+3n-8}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{6n^7-6n^6+5n^5}{12n^7+3n-8} = \frac{6}{12} = \frac{1}{2}$$ finite and positive

then $\sum_{n=1}^{\infty} \dfrac{6n^2-6n+5}{12n^7+3n-8}$ converges

(b) Ratio test

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{\frac{\ln(k+1)}{9e^{k+1}}}{\frac{\ln k}{9e^k}} = \lim_{k\to\infty} \frac{\ln(k+1)}{e \ln k}$$ indeterminate form,

applying L'Hôpital's rule, $\lim_{k\to\infty} \dfrac{\ln(k+1)}{e\ln k} = \lim_{k\to\infty} \dfrac{\frac{1}{k+1}}{e\frac{1}{k}} = \lim_{k\to\infty} \dfrac{k}{e(k+1)} = \dfrac{1}{e} < 1$

then $\sum_{k=1}^{\infty} \dfrac{\ln k}{9e^k}$ converges

18. Ratio test, $\lim_{k\to\infty} \left| \dfrac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \dfrac{\frac{x^{k+1}}{k+5}}{\frac{x^k}{k+4}} \right| = \lim_{k\to\infty} \dfrac{k+4}{k+5} |x| = |x|$

由 $|x| < 1$，得 the radius of convergence $R = 1$

At x = -1, $\sum_{k=4}^{\infty} \dfrac{(-1)^k}{k}$ converges (alternating series test)

At x = 1, $\sum_{k=4}^{\infty} \dfrac{1}{k}$ diverges (harmonic series)

The interval of convergence is $[-1, 1)$

19. $\vec{r}'(t) = 3t^2\,\hat{i} + \hat{j} + \sqrt{6}t\,\hat{k}$, $\quad \|\vec{r}'(t)\| = \sqrt{9t^4 + 1 + 6t^2} = \sqrt{(3t^2+1)^2} = 3t^2 + 1$

$$s = \int_2^5 \|\vec{r}'(t)\| dt = \int_2^5 (3t^2+1)dt = t^3 + t\Big|_2^5 = 130 - 10 = \underline{120}$$

20. Let $F(x,y,z) = \sin(xy+z) - x - y$, $\quad F_x = y\cos(xy+z) - 1$, $\quad F_z = \cos(xy+z)$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{1 - y\cos(xy+z)}{\cos(xy+z)}$$

21. $T = x^2 + y^3$, $\quad x^2 + y^2 = 1$, $\quad$ let $x = \cos\theta$, $\quad y = \sin\theta$, $\quad 0 \le \theta \le \pi$

$$T = \cos^2\theta + \sin^3\theta,\quad \frac{dT}{d\theta} = 2\cos\theta(-\sin\theta) + 3\sin^2\theta\cos\theta = \sin\theta\cos\theta(3\sin\theta - 2)$$

$$\frac{dT}{d\theta} = 0 \quad\Rightarrow\quad \sin\theta = 0,\quad \cos\theta = 0,\quad \text{or}\quad \sin\theta = \frac{2}{3}$$

(1) $\sin\theta = 0, \quad\Rightarrow\quad \theta = 0, \quad \theta = \pi \quad\Rightarrow\quad \cos\theta = \pm 1 \quad\Rightarrow\quad T = 1$

(2) $\cos\theta = 0, \quad\Rightarrow\quad \theta = \frac{\pi}{2} \quad\Rightarrow\quad \sin\theta = 1 \quad\Rightarrow\quad T = 1$

(3) $\sin\theta = \frac{2}{3}, \quad 0 \le \theta \le \pi \quad\Rightarrow\quad \cos\theta = \pm\frac{\sqrt{5}}{3} \quad\Rightarrow\quad T = \frac{5}{9} + \frac{8}{27} = \frac{23}{27}$ $\quad$ (a) Minimum

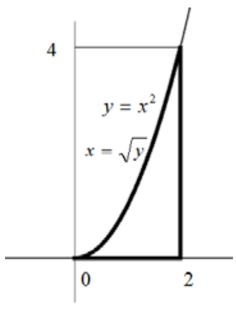
$$\frac{dT}{dt} = \frac{dT}{d\theta}\frac{d\theta}{dt} = \sin\theta\cos\theta(3\sin\theta - 2)\frac{d\theta}{dt} = \sin\theta\cos\theta(3\sin\theta - 2) \qquad \because \quad \frac{d\theta}{dt} = \frac{2\pi}{2\pi} = 1$$

$$\theta = \frac{\pi}{4},\quad \frac{dT}{dt}\Big|_{\theta=\pi/4} = \sin\frac{\pi}{4}\cos\frac{\pi}{4}(3\sin\frac{\pi}{4} - 2) = \frac{1}{2}(\frac{3}{\sqrt{2}} - 2) = \frac{3}{2\sqrt{2}} - 1 \quad \text{(b)}$$

22. $\displaystyle\int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x} + C$

$$\int_1^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}dx = -\lim_{b\to\infty}[\frac{\ln x}{x} + \frac{1}{x}]_1^b = -\lim_{b\to\infty}[\frac{\ln b}{b} + \frac{1}{b} - 0 - 1] = -(0 + 0 - 1) = 1$$

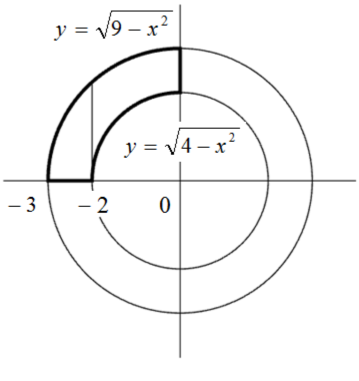
23. $\displaystyle\int_0^4\int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dxdy = \int_0^2\int_0^{x^2} \sqrt{x^3+1}\,dydx$



$$= \int_0^2 x^2\sqrt{x^3+1}\,dx \qquad \text{let}\quad z = x^3 + 1$$

$$= \frac{1}{3}\int_1^9 \sqrt{z}dz = \frac{1}{3}\cdot\frac{2}{3}[z^{3/2}]_1^9 = \frac{2}{9}(27 - 1) = \frac{52}{9}$$

24. $\displaystyle\int_{-3}^{-2}\int_0^{\sqrt{9-x^2}} (x^2+y^2)dydx + \int_{-2}^{0}\int_{\sqrt{4-x^2}}^{\sqrt{9-x^2}} (x^2+y^2)dydx$



$$= \int_{\pi/2}^{\pi}\int_2^3 r^2\cdot r dr d\theta = \int_{\pi/2}^{\pi}[\frac{1}{4}r^4]_2^3\,d\theta = \frac{\pi}{2}\cdot\frac{1}{4}(81 - 16)$$

$$= \frac{65\pi}{8}$$