

國立宜蘭大學 110 年度微積分競試 解答

1. 收斂	2. 發散
3. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$	4. $\langle 1, 12, 0 \rangle$
5. 1	6. -3
7. e^{-1}	8. 1
9. $-2\theta \sin(\theta^2)$	10. $\frac{8(1-\cot x - x \csc^2 x)}{(1-\cot x)^2}$
11. $\frac{x-4y}{4x-y}$	12. $\sqrt{17}$
13. $\ln \left \frac{x+2}{x+3} \right + C$	14. $\frac{1}{2\ln 2} \ln 2^{2x} - 2^{-2x} + C$
15. $\frac{1}{\ln ab} (ab)^x + C$	16. $\frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C$
17. 147	18. $2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}$
19. -86.11	20. $\langle (e^2 - 1) \cos(6), -(e^2 - 2) \sin(6) \rangle$
21. $e^4 - 1$	22. $-\frac{1}{3}$ or $\frac{1}{2}$

1.

$$\sum_{n=0}^{\infty} \frac{\sin(n+\frac{1}{2})\pi}{1+d^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+d^n}, \quad a_n = \frac{1}{1+d^n} > 0 \text{ for } n \geq 0$$

$$a_{n+1} = \frac{1}{1+d^{n+1}} < \frac{1}{1+d^n} = a_n, \quad \& \lim_{n \rightarrow \infty} \frac{1}{1+d^n} = 0$$

2. con. alternating series test

$$b_n = \frac{1}{n}, \quad \sum \frac{1}{n}, \quad p=1 \text{ div. } \text{ (做 } \underline{\text{limit comparison}}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{ne^n + n}{ne^n + 1} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{e^n}}{1 + \frac{1}{ne^n}} = 1 \in \mathbb{R}$$

∴ div.

3.

$$\cos x \stackrel{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow$$

$$f(x) = \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

$$= 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8 - \frac{1}{720}x^{12} + \dots$$

4.

$$f(x, y, z) = xe^{2yz}$$

$$(a) \nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle = \langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz} \rangle$$

$$(b) \nabla f(3, 0, 2) = \langle 1, 12, 0 \rangle$$

5.

$$\text{that } f(x) = \frac{3}{2x + f(x)}$$

Assumed that the denominator " $2x + f(x)$ " is not zero, then we have

$$[f(x)]^2 + (2x)f(x) - 3 = 0$$

$$\Rightarrow f(x) = -x + \sqrt{x^2 + 3}$$

$$\text{or } f(x) = -x - \sqrt{x^2 + 3}$$

however,

$$f(1) = \frac{3}{2 + \frac{3}{2 + \frac{3}{\vdots}}}$$

$\Rightarrow f(1)$ is positive.

$$\Rightarrow f(1) = 1$$

6.

$$\begin{aligned} & \lim_{x \rightarrow \infty} \sqrt{x(x-3)} - \sqrt{x(x+3)} \\ &= \lim_{x \rightarrow \infty} \left[\sqrt{x(x-3)} - \sqrt{x(x+3)} \right] \cdot \left[\frac{\sqrt{x(x-3)} + \sqrt{x(x+3)}}{\sqrt{x(x-3)} + \sqrt{x(x+3)}} \right] \\ &= \lim_{x \rightarrow \infty} \frac{x(x-3) - x(x+3)}{\sqrt{x(x-3)} + \sqrt{x(x+3)}} \\ &= -6 \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x(x-3)} + \sqrt{x(x+3)}} \end{aligned}$$

$$\begin{aligned} &= -6 \lim_{x \rightarrow \infty} \frac{\frac{x}{x}}{\frac{\sqrt{x(x-3)} + \sqrt{x(x+3)}}{x}} \\ &= -6 \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 - \frac{3}{x}} + \sqrt{1 + \frac{3}{x}}} \\ &= -6 \lim_{x \rightarrow \infty} \frac{1}{1+1} \\ &= -3 \end{aligned}$$

7.

$$\begin{aligned} & \lim_{x \rightarrow 1^+} x^{\frac{1}{1-x}} \\ \text{Let } y &= \lim_{x \rightarrow 1^+} x^{\frac{1}{1-x}} \\ \ln y &= \ln \lim_{x \rightarrow 1^+} x^{\frac{1}{1-x}} \\ &= \lim_{x \rightarrow 1^+} \ln x^{\frac{1}{1-x}} \\ \ln x^{\frac{1}{1-x}} &= \frac{1}{1-x} \ln x = \frac{\ln x}{1-x} \\ \lim_{x \rightarrow 1^+} \frac{\ln x}{1-x} &\Rightarrow \frac{0}{0} \text{ type.} \end{aligned}$$

L'Hôpital's rule can be applied here.

$$\begin{aligned} &\Rightarrow \lim_{x \rightarrow 1^+} \frac{\frac{1}{x}}{-1} \\ &= -1 \\ &\Rightarrow \ln y = -1 \\ & y = e^{-1} \\ \therefore \lim_{x \rightarrow 1^+} x^{\frac{1}{1-x}} &= e^{-1} \end{aligned}$$

8.

$$\begin{aligned} & \lim_{x \rightarrow 0^+} x \tan\left(\frac{\pi}{2} - x\right) \\ &= \lim_{x \rightarrow 0^+} x \cdot \left[\frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} \right] \\ &\Rightarrow 0 \cdot \frac{1}{0} \Rightarrow \frac{0}{0} \text{ type} \\ & \text{L'Hôpital's rule can be applied.} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \lim_{x \rightarrow 0^+} \frac{\sin\left(\frac{\pi}{2} - x\right) - x \cdot [\cos\left(\frac{\pi}{2} - x\right)](-1)}{[-\sin\left(\frac{\pi}{2} - x\right)](-1)} \\ &= \frac{1 - 0 \cdot 0 \cdot (-1)}{1} = 1 \\ \therefore \lim_{x \rightarrow 0^+} x \tan\left(\frac{\pi}{2} - x\right) &= 1 \end{aligned}$$

9.

$$\frac{d f(\theta)}{d \theta} = \frac{d \cos(\theta^2)}{d \theta} = (-) \sin(\theta^2) \cdot 2\theta$$

10.

$$\begin{aligned}
 y' &= \frac{d}{dx} \left(\frac{8x}{1 - \cot(x)} \right) \\
 &= \frac{8 \cdot (1 - \cot(x)) - 8x \cdot (-)(-) \csc^2(x)}{[1 - \cot(x)]^2} \\
 &= \frac{8(1 - \cot(x) - x \cdot \csc^2(x))}{[1 - \cot(x)]^2}
 \end{aligned}$$

11.

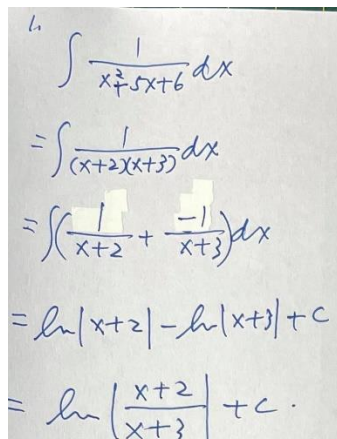
$$\begin{aligned}
 \frac{d}{dx}(x^2 - 8xy + y^2) &= 8 \\
 2x - 8y - 8x \cdot y' + 2y \cdot y' &= 0 \\
 y' &= \frac{2x - 8y}{8x - 2y} = \frac{x - 4y}{4x - y}
 \end{aligned}$$

12.

$$\begin{aligned}
 \nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \\
 f_x(x, y) &= \tan y & \nabla f(x, y) &= \tan y\mathbf{i} + x \sec^2 y\mathbf{j} \\
 f_y(x, y) &= x \sec^2 y & \nabla f\left(2, \frac{\pi}{4}\right) &= \tan \frac{\pi}{4}\mathbf{i} + 2 \sec^2 \frac{\pi}{4}\mathbf{j} \\
 & & &= \mathbf{i} + 4\mathbf{j}
 \end{aligned}$$

The maximum value of directional derivative is equal to $\|\nabla f(2, \frac{\pi}{4})\| = \sqrt{1 + 4^2} = \sqrt{17}$

13.



$$\begin{aligned}
 &\int \frac{1}{x^2 + 5x + 6} dx \\
 &= \int \frac{1}{(x+2)(x+3)} dx \\
 &= \int \left(\frac{1}{x+2} + \frac{-1}{x+3} \right) dx \\
 &= \ln|x+2| - \ln|x+3| + C \\
 &= \ln \left| \frac{x+2}{x+3} \right| + C
 \end{aligned}$$

14.

$$2. \int \frac{2^{2^x} + 2^{-2^x}}{2^{2^x} - 2^{-2^x}} dx$$

$$u = 2^{2^x} - 2^{-2^x}$$

$$du = 2(2^{2^x} + 2^{-2^x}) \ln 2$$

$$\int \frac{1}{u} du = \int \frac{1}{2 \ln 2} du$$

$$= \frac{1}{2 \ln 2} \ln |2^{2^x} - 2^{-2^x}| + C$$

15.

$$3. \int a^x b^x dx$$

$$= \int (ab)^x dx$$

$$= \frac{1}{\ln ab} (ab)^x + C$$

16.

$$6. F'(x) = x^2 \ln x$$

$$F(x) = \int F'(x) dx$$

$$= \int x^2 \ln x dx$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C$$

17.

$$\text{令 } g(x, y, z) = 2x - 3y - 4z$$

$$f(x, y, z) = 2x^2 + y^2 + 3z^2$$

$$4x = 2\lambda$$

$$2y = -3\lambda$$

$$\nabla f(x, y, z) = 4x\mathbf{i} + 2y\mathbf{j} + 6z\mathbf{k}$$

$$6z = -4\lambda$$

$$\lambda \nabla g = 2\lambda\mathbf{i} - 3\lambda\mathbf{j} - 4\lambda\mathbf{k}$$

$$2x - 3y - 4z = 49$$

$$f(3, -9, -4) = 2(3)^2 + (-9)^2 + 3(-4)^2$$

$$\text{解得 } x = 3, y = -9, z = -4, \lambda = 6$$

$$= 147$$

18.

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= 2xy (\cos t) + (x^2 - 2y) e^t \\ &= 2(\sin t) (e^t) (\cos t) + (\sin^2 t - 2e^t) e^t \\ &= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}\end{aligned}$$

19.

$$\begin{aligned}\frac{1}{1+x^3} &= \sum_{n=0}^{\infty} (-x^3)^n \approx 1 - x^3 + x^6 - x^9 \rightarrow \\ \int_0^2 \frac{1}{1+x^3} dx &\approx \int_0^2 1 - x^3 + x^6 - x^9 dx = -86.11\end{aligned}$$

20.

$$\begin{aligned}\nabla f &= \langle (e^x - 1) \cos(y), -(e^x - x) \sin(y) \rangle \rightarrow \\ \nabla f(2,6) &= \langle (e^2 - 1) \cos(6), -(e^2 - 2) \sin(6) \rangle\end{aligned}$$

21.

$$\begin{aligned}\int_0^1 \int_{2x}^2 4e^{y^2} dy dx &= 4 \int_0^2 \int_0^{\frac{1}{2}y} e^{y^2} dx dy \\ &= 4 \int_0^2 \left[x e^{y^2} \right]_0^{\frac{1}{2}y} dy \\ &= 4 \int_0^2 \left(\frac{1}{2} y e^{y^2} \right) dy \\ &= \int_0^2 (2y e^{y^2}) dy \\ &= \left[e^{y^2} \right]_0^2 \\ &= e^4 - 1\end{aligned}$$

22.

$$\operatorname{div} \vec{F} = 6a^2 - a - 1 = 0 \rightarrow a = -\frac{1}{3} \text{ or } \frac{1}{2}$$